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### MODULE PREPARATION

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Solutions

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- Constructing matrices

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- Addition and subtraction of matrices
- Using CAS: Adding and subtracting matrices
- Scalar multiplication
- Using CAS: Addition, subtraction and scalar multiplication of matrices

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- Solving two simultaneous equations using matrices
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- Solving three or more simultaneous equations using matrices
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7.7 Communication matrices
- Communication diagrams and matrices
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Summary
Welcome to the matrix

What do databases, costing and pricing, quantum mechanics, and links in communication, social and road networks have in common? They can all be modelled by matrices.

A matrix is a rectangular arrangement of numbers organised into rows and columns.

Matrices are usually presented in square brackets, for example:

\[
\begin{pmatrix}
3 & 2 & 4 \\
1 & -2 & 4 \\
\end{pmatrix}

\begin{pmatrix}
0 & -1 \\
5 & 13 \\
\end{pmatrix}

\begin{pmatrix}
21 & 15 & -42 & 11 \\
\end{pmatrix}
\]

The table and matrix below show how senior students get to their school each day.

<table>
<thead>
<tr>
<th>Senior Students’ Methods of Travel to School</th>
<th>Matrix equivalent</th>
</tr>
</thead>
</table>
| Year 11 | Year 12 | \( A = \begin{bmatrix}
32 & 18 \\
78 & 38 \\
27 & 12 \\
9 & 31 \\
\end{bmatrix} \) |
| Walk    | 32     | 18     |
| Bus     | 78     | 38     |
| Car (other driver) | 27 | 12 |
| Car (self-driven)   | 9     | 31     |

We usually name matrices using capital letters. Each value in a matrix is called an element. Matrix \( A \) has 8 elements. There are four rows in matrix \( A \). The first row in matrix \( A \) contains the elements 32 and 18. There are two columns in matrix \( A \). The second column contains the elements 18, 38, 12 and 31.
Types of matrices

The order of a matrix tells us how many rows and columns it has. For example, the order of matrix $A$ on the previous page is ‘four by two’ or $4 \times 2$. The order of a matrix also tells us how many elements it has. For example, a $4 \times 2$ matrix has 8 elements, and an $11 \times 3$ matrix has 33 elements.

The order of a matrix is written as $m \times n$, and we say ‘$m$ by $n$’, where $m$ is the number of rows and $n$ is the number of columns.

An $m \times n$ matrix has $mn$ elements.

<table>
<thead>
<tr>
<th>Type of matrix</th>
<th>Description</th>
<th>Example</th>
<th>Order of example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row matrix</td>
<td>A matrix with just one row.</td>
<td>$\begin{bmatrix} -2 &amp; 3 &amp; 11 &amp; 5 \end{bmatrix}$</td>
<td>$1 \times 4$</td>
</tr>
<tr>
<td>Column matrix</td>
<td>A matrix with just one column.</td>
<td>$\begin{bmatrix} 12 \ -1 \ 0 \ 5 \end{bmatrix}$</td>
<td>$4 \times 1$</td>
</tr>
<tr>
<td>Zero matrix</td>
<td>A matrix where all the elements are 0.</td>
<td>$\begin{bmatrix} 0 &amp; 0 \ 0 &amp; 0 \ 0 &amp; 0 \ 0 &amp; 0 \end{bmatrix}$</td>
<td>$2 \times 2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{bmatrix}$</td>
<td>$2 \times 3$</td>
</tr>
<tr>
<td>Square matrix</td>
<td>A matrix that has the same number of rows as columns.</td>
<td>$\begin{bmatrix} 0 &amp; 9 \ 5 &amp; 3 \ 2 &amp; 2 &amp; 6 \ 7 &amp; 1 &amp; 0 \ 1 &amp; 0 &amp; 10 \end{bmatrix}$</td>
<td>$2 \times 2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{bmatrix} 4 &amp; 0 \ 0 &amp; -7 \ 4 &amp; 0 &amp; 0 \ 0 &amp; -8 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{bmatrix}$</td>
<td>$3 \times 3$</td>
</tr>
<tr>
<td>Diagonal matrix</td>
<td>A square matrix where the only non-zero elements are in the leading diagonal (the diagonal running from the upper left to the lower right).</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Identity matrix</td>
<td>A square matrix where all the elements in the leading diagonal are 1 and the other elements are 0. We use $I$ to denote the identity matrix.</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; 1 \ 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{bmatrix}$</td>
<td>$2 \times 2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; 1 \ 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{bmatrix}$</td>
<td>$3 \times 3$</td>
</tr>
</tbody>
</table>
Tickets to a school production of *Fame* have been sold in three different ways as shown in the table. Write each matrix described.

<table>
<thead>
<tr>
<th></th>
<th>Sold by school office</th>
<th>Sold online</th>
<th>Sold at theatre</th>
</tr>
</thead>
<tbody>
<tr>
<td>Student ticket</td>
<td>183</td>
<td>78</td>
<td>41</td>
</tr>
<tr>
<td>Adult ticket</td>
<td>4</td>
<td>140</td>
<td>5</td>
</tr>
<tr>
<td>Concession ticket</td>
<td>0</td>
<td>54</td>
<td>7</td>
</tr>
</tbody>
</table>

**a** The matrix, $T$, that could be used to show this information and state the order of the matrix.

**b** The row matrix that could be used to show the numbers of concession tickets sold in each of the different ways and state the order of the matrix.

**c** The $3 \times 1$ matrix that could be used to show the number of student tickets sold in each of the different ways.

**d** The $1 \times 3$ matrix that could be used to show the number of each type of ticket sold online.

**e** The column matrix that could be used to show the total number of each type of ticket sold.

**f** The column matrix that could be used to show the total number of each of the ways the tickets have been sold.

### Working

**a** Rewrite the information in the table as a matrix.

\[
T = \begin{bmatrix}
183 & 78 & 41 \\
4 & 140 & 5 \\
0 & 54 & 7
\end{bmatrix}
\]

The order of $T$ is $3 \times 3$.

**b** Find the information in the table and write in the correct matrix form.

\[
\begin{bmatrix}
0 & 54 & 7
\end{bmatrix}
\]

The order of the matrix is $1 \times 3$.

**c** Find the information in the table and write in the correct matrix form.

\[
\begin{bmatrix}
183 \\
78 \\
41
\end{bmatrix}
\]

**d** Find the information in the table and write in the correct matrix form.

\[
\begin{bmatrix}
78 & 140 & 54
\end{bmatrix}
\]

**e** Add the appropriate table entries and write in the correct matrix form.

\[
\begin{bmatrix}
183 + 78 + 41 \\
4 + 140 + 5 \\
0 + 54 + 7
\end{bmatrix}
= \begin{bmatrix}
302 \\
149 \\
61
\end{bmatrix}
\]

**f** Add the appropriate table entries and write in the correct matrix form.

\[
\begin{bmatrix}
183 + 4 + 0 \\
78 + 140 + 54 \\
41 + 5 + 7
\end{bmatrix}
= \begin{bmatrix}
187 \\
272 \\
53
\end{bmatrix}
\]
Constructing matrices

The position of each element in a matrix is indicated by referring to which row and column it is situated in.

For example, in the matrix

\[
A = \begin{bmatrix}
5 & 2 \\
8 & 9 \\
3 & 7 \\
\end{bmatrix}
\]

\(a_{11}\) is the element in row 1 and column 1 \((a_{11} = 5)\)

\(a_{12}\) is the element in row 1 and column 2 \((a_{12} = 2)\)

\(a_{21} = 8, a_{22} = 9, a_{31} = 3, a_{32} = 7\)

\(a_{ij}\) is an element of matrix \(A\) where \(i\) is the row number and \(j\) is the column number.

For example, the elements of a 3 × 3 matrix can be written as shown.

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33} \\
\end{bmatrix}
\]

Worked example 2

Construct the matrix, \(A\), for each of the following rules about its elements, \(a_{ij}\).

\(\textbf{a}\) \(a_{11} = 3, a_{12} = 9, a_{21} = -2, a_{22} = 6\)

\(\textbf{b}\) \(a_{11} = 2, a_{12} = 10, a_{13} = 5, a_{14} = 7\)

\(\textbf{c}\) \(A\) is a 3 × 2 matrix where \(a_{12} = 6, a_{21} = 5, a_{31} = 9, a_{32} = 7,\) and \(a_{ij} = 2\) when \(i = j\).

\textbf{Working}

\(\textbf{a}\) Use the definition for \(a_{ij}\), where \(i\) is the row number and \(j\) is the column number, to construct the matrix.

\[
A = \begin{bmatrix}
3 & 9 \\
-2 & 6 \\
\end{bmatrix}
\]

\(\textbf{b}\)

\[
A = \begin{bmatrix}
2 & 10 & 5 & 7 \\
\end{bmatrix}
\]

\(\textbf{c}\) \(a_{ij} = 2\) when \(i = j\) means \(a_{11} = a_{22} = 2\).

\[
A = \begin{bmatrix}
2 & 6 \\
5 & 2 \\
9 & 7 \\
\end{bmatrix}
\]
Introduction to matrices

Prep 1

a

\[
\begin{bmatrix}
-5 & 0 \\
8 & 0
\end{bmatrix}
\]

Is this a column, square or zero matrix? What is its order?

b

\[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

Is this a diagonal, identity or zero matrix? What is its order?

c

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Is this a zero, identity or column matrix? What is its order?

Prep 2

Answer true or false to each of the following statements.

a A 4 × 3 matrix has 43 elements.

b A 2 × 10 matrix has 20 elements.

c An identity matrix with five 1s has 25 elements.

d If a matrix has 5 elements, it has to be either a row matrix or a column matrix.

Prep 3

WORKED EXAMPLE 1

The table shows the quantity (in grams) of the main ingredients used to bake various cakes.

<table>
<thead>
<tr>
<th>Type of cake</th>
<th>Ingredient</th>
<th>Chocolate cake</th>
<th>Fruit cake</th>
<th>Tea cake</th>
<th>Banana cake</th>
<th>Butter cake</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sugar</td>
<td>100</td>
<td>80</td>
<td>80</td>
<td>75</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>Flour</td>
<td>225</td>
<td>125</td>
<td>80</td>
<td>150</td>
<td>175</td>
<td>150</td>
</tr>
<tr>
<td>Butter</td>
<td>125</td>
<td>100</td>
<td>150</td>
<td>150</td>
<td>175</td>
<td></td>
</tr>
</tbody>
</table>

a Write the matrix, \( T \), that could be used to show this information and state its order.

b Write the row matrix that could be used to show the amount of flour needed in each of the cakes and state its order.

c Write the 5 × 1 matrix that could be used to show the amount of sugar needed in each cake.

d Write the 1 × 3 matrix that could be used to show the amount of each ingredient needed in a banana cake.
e Write the column matrix that could be used to show the total amounts of each ingredient needed if one of every type of cake is made.

f Write the column matrix that could be used to show the total number of grams of ingredients in each of the cakes.

**Worked Example 2**

Construct the matrix, A, for each of the following rules about the elements, $a_{ij}$.

- **a** $a_{11} = 4, a_{21} = -5, a_{31} = 1, a_{41} = 7$
- **b** $a_{11} = -1, a_{12} = 0, a_{21} = 6, a_{22} = 3$
- **c** $A$ is a $2 \times 3$ matrix where $a_{13} = 8, a_{12} = 1, a_{21} = 5, a_{23} = 7$, and $a_{ij} = 1$ when $i = j$.

**Introduction to matrices**

**Question 1**

Consider the following arrays.

\[ J = \begin{bmatrix} 3 & 5 \\ 6 & 2 \end{bmatrix} \quad K = \begin{bmatrix} -2 \\ 5 \\ 1.1 \end{bmatrix} \quad L = \begin{bmatrix} 2 & 1.6 & 3 \\ 4 & 1 & -2 \\ 5 \end{bmatrix} \quad M = \begin{bmatrix} 12 \end{bmatrix} \]

Which statement is true?

- **A** None of these arrays are matrices.
- **B** $J$ and $K$ are matrices, but $L$ and $M$ are not.
- **C** All of these arrays are matrices.
- **D** $M$ is the only array that is not a matrix.
- **E** $L$ is the only array that is not a matrix.

**Question 2**

Matrix $T$ has 4 rows and 3 columns. How many elements does it have?

- **A** 4
- **B** 12
- **C** 3
- **D** 7
- **E** 16

**Question 3**

Given that matrix $P = \begin{bmatrix} 5 & 6 & 3 & 7 \\ 2 & 10 & 2 & 1 \\ 5 & 0 & 9 & 7 \end{bmatrix}$, $p_{23} + p_{31}$ equals

- **A** 7
- **B** 3
- **C** 5
- **D** 12
- **E** 9
Question 4
What type of matrix is \[\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\]?

A  A zero matrix   B  An identity matrix   C  A 3 \times 2 matrix
D  A square matrix   E  A 2 \times 3 matrix

Question 5
Which of the following is incorrect about \[M = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix}\]?

A  \(M\) is a 3 \times 3 matrix   B  \(M\) is a column matrix   C  \(M\) is a square matrix
D  \(M\) has 9 elements   E  \(M\) is a diagonal matrix

Question 6
The orders of the matrices that could be used to represent the information in the table are

A  6 \times 2 or 2 \times 6   B  6 \times 2 only
C  2 \times 6 only   D  7 \times 3 or 3 \times 7
E  7 \times 3 only

Senior Students’ Favourite Takeaway Food

<table>
<thead>
<tr>
<th></th>
<th>Year 11</th>
<th>Year 12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pizza</td>
<td>32</td>
<td>15</td>
</tr>
<tr>
<td>Chicken</td>
<td>38</td>
<td>27</td>
</tr>
<tr>
<td>Fish and chips</td>
<td>30</td>
<td>33</td>
</tr>
<tr>
<td>Hamburgers</td>
<td>9</td>
<td>32</td>
</tr>
<tr>
<td>Kebabs</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Other</td>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

Question 7
For the matrix \[A = \begin{bmatrix} 2 & 6 & 4 & 8 \\ 2 & 0 & 9 & 2 \\ 7 & 8 & 9 & 1 \\ 9 & 0 & 6 & 7 \end{bmatrix}\], which is the element \(a_{34}\)?

A  3   B  6   C  1   D  4   E  7
Question 8

If \( K = \begin{bmatrix} 9 & 12 & 0 \\ 3 & 15 & 2 \end{bmatrix} \), what is the value of \( k_{13} + k_{11} \)?

A 12  B 9  C 3  D 5  E 18

Question 9

Which one of the following rules could be used to construct the matrix \( A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \)?

A \( a_{12} = 2, a_{21} = -1, a_{11} = -3, a_{22} = 2 \)
B \( a_{11} = 2, a_{12} = -3, a_{21} = -1, a_{22} = 2 \)
C \( a_{12} = -3, a_{21} = -1, \text{ and } a_{ij} = 2 \text{ when } i = j. \)
D \( a_{12} = -1, a_{21} = -3, \text{ and } a_{ij} = 2 \text{ when } i = j. \)
E \( a_{11} = 2, a_{12} = -1, a_{13} = -1, a_{14} = 2 \)

Question 10

The matrix shows the airfares (in dollars) that are charged by Zeniff Airlines to fly between Adelaide (A), Melbourne (M) and Sydney (S).

The cost to fly from Melbourne to Sydney with Zeniff Airlines is

A $85  B $89  C $97
D $99  E $101

[VCAA 2011 1MQ1]

Question 11

The number of tourists visiting three towns, Oldtown, Newtown and Twixtown, was recorded for three years. The data is summarised in the table.

The 3 × 1 matrix that could be used to show the number of tourists visiting the three towns in the year 2005 is

A \( \begin{bmatrix} 975 & 1002 & 1390 \end{bmatrix} \)
B \( \begin{bmatrix} 1002 & 1081 & 1095 \end{bmatrix} \)
C \( \begin{bmatrix} 975 \\ 1002 \\ 1390 \end{bmatrix} \)
D \( \begin{bmatrix} 1002 \\ 1081 \\ 1095 \end{bmatrix} \)
E \( \begin{bmatrix} 975 & 1002 & 1390 \\ 2105 & 1081 & 1228 \\ 610 & 1095 & 1380 \end{bmatrix} \)

[VCAA 2007 1MQ2]
Question 12

The order of the matrix \[
\begin{bmatrix}
2 & 2 \\
2 & 2 \\
2 & 2
\end{bmatrix}
\] is

A. \(2 \times 2\)  
B. \(2 \times 3\)  
C. \(3 \times 2\)  
D. \(4\)  
E. \(6\)

[VCAA 2010 1MQ1]

Question 13

The number of people attending the morning, afternoon and evening sessions at a cinema is given in the table. The admission charges (in dollars) for each session are also shown in the table.

<table>
<thead>
<tr>
<th>Session</th>
<th>Morning</th>
<th>Afternoon</th>
<th>Evening</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of people attending</td>
<td>25</td>
<td>56</td>
<td>124</td>
</tr>
<tr>
<td>Admission charge ($)</td>
<td>12</td>
<td>15</td>
<td>20</td>
</tr>
</tbody>
</table>

A column matrix that can be used to list the number of people attending each of the three sessions is

\[
\begin{bmatrix}
25 & 56 & 124
\end{bmatrix}
\]

A  
B  
C  
D  
E

[VCAA 2009 1MQ3]
Addition and subtraction of matrices

Addition and subtraction of matrices can only be performed with matrices of the same order. Addition or subtraction of matrices involves adding or subtracting the elements which are in corresponding positions in both matrices.

So, if we have matrix $A = \begin{bmatrix} 2 & 7 \\ 11 & 3 \end{bmatrix}$ and matrix $B = \begin{bmatrix} 1 & 3 \\ 5 & 12 \end{bmatrix}$,

then $A + B = \begin{bmatrix} 2+1 & 7+3 \\ 11+5 & 3+12 \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ 16 & 15 \end{bmatrix}$ and $A - B = \begin{bmatrix} 2-1 & 7-3 \\ 11-5 & 3-12 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 6 & -9 \end{bmatrix}$

Worked example 3

If $D = \begin{bmatrix} 5 & 6 & 1 \\ 14 & -7 & 2 \end{bmatrix}$ and $E = \begin{bmatrix} -3 & 0 & 4 \\ 11 & 5 & 10 \end{bmatrix}$, find

$\textbf{a} \quad D + E \quad \textbf{b} \quad D - E$

Working

$D + E = \begin{bmatrix} 5+(-3) & 6+0 & 1+4 \\ 14+11 & -7+5 & 2+10 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 5 \\ 25 & -2 & 12 \end{bmatrix}$

$D - E = \begin{bmatrix} 5-(-3) & 6-0 & 1-4 \\ 14-11 & -7-5 & 2-10 \end{bmatrix} = \begin{bmatrix} 8 & 6 & -3 \\ 3 & -12 & -8 \end{bmatrix}$
Using CAS  Adding and subtracting matrices

Use a CAS/calculator to evaluate:
\[
\begin{bmatrix}
12 & 3 \\
18 & 5
\end{bmatrix} + \begin{bmatrix}
13 & -9 \\
6 & 22
\end{bmatrix}
\]

**TI-Nspire CAS**

**STEP 1**
Use a New Document with a Calculator page.
Press \( \text{[ } \) .
Select the 2 \( \times \) 2 matrix template \( \begin{bmatrix}
\square & \square \\
\square & \square
\end{bmatrix} \) and press \( \text{[ enter]} \).

**STEP 2**
Enter the elements, pressing \( \text{[ tab]} \) after each entry.

**STEP 3**
Press \( + \), then create a second 2 \( \times \) 2 matrix using the method outlined above.
Enter the elements of the second matrix.
Press \( \text{[ enter]} \).

**Classpad**

**STEP 1**
Use the \( \text{M} \) application.
Press \( \text{[ Function]} \).
Tap \( \text{[ Math]} \), then tap the 2 \( \times \) 2 matrix template \( \begin{bmatrix}
\square & \square \\
\square & \square
\end{bmatrix} \).

**STEP 2**
Enter the elements, using the arrow keys \( \text{[ up]} \) \( \text{[ down]} \) \( \text{[ left]} \) \( \text{[ right]} \) to position the cursor in the correct position for each entry.
STEP 3

Press \( \text{ } \) as necessary to move the cursor outside of the first matrix. Press \( + \), then create a second \( 2 \times 2 \) matrix using the method outlined above.

Enter the elements of the second matrix.

Press \( \text{EXE} \).

### Worked example 4

Find the value of \( x \), \( y \) and \( z \) in the following:

\[
\begin{bmatrix}
6 & 12 \\
y & 7 \\
\end{bmatrix} + \begin{bmatrix}
6 & -5 \\
1 & 2z \\
\end{bmatrix} = \begin{bmatrix}
0 & x \\
4 & -3 \\
\end{bmatrix}
\]

#### Working

1. Using the corresponding elements in row 1 and column 2 of each matrix, write down an equation. 
\( 12 - (-5) = x \)

2. Solve for \( x \), using a CAS/calculator if necessary. 
\( x = 17 \)

3. Using the corresponding elements in row 2 and column 1 of each matrix, write down an equation. 
\( y - 1 = 4 \)

4. Solve for \( y \), using a CAS/calculator if necessary. 
\( y = 5 \)

5. Using the corresponding elements in row 2 and column 2 of each matrix, write down an equation. 
\( 7 - 2z = -3 \)

6. Solve for \( z \), using a CAS/calculator if necessary. 
\( 2z = 10 \)
\( z = 5 \)

### Scalar multiplication

**Scalar multiplication** involves multiplying a matrix by a number. When dealing with matrices, we use the word **scalar** to indicate a number that’s not in a matrix.

To multiply matrix \( A \) by the scalar 4, we simply multiply each of the elements in matrix \( A \) by 4.

\[
\text{If } A = \begin{bmatrix}
5 & 2 \\
7 & -3 \\
12 & 6 \\
\end{bmatrix}, \text{ then } 4A = 4 \begin{bmatrix}
5 & 2 \\
7 & -3 \\
12 & 6 \\
\end{bmatrix} = \begin{bmatrix}
4 \times 5 & 4 \times 2 \\
4 \times 7 & 4 \times (-3) \\
4 \times 12 & 4 \times 6 \\
\end{bmatrix} = \begin{bmatrix}
20 & 8 \\
28 & -12 \\
48 & 24 \\
\end{bmatrix}
\]
Using CAS  

Addition, subtraction and scalar multiplication of matrices

Given that \( Q = \begin{bmatrix} 3 & -5 & 10 \\ 11 & 6 & 17 \\ -2 & 15 & 0 \end{bmatrix} \) and \( R = \begin{bmatrix} 12 & 2 & 1 \\ -3 & 4 & 7 \\ 22 & 14 & -7 \end{bmatrix} \), evaluate \( 4Q - 2R \) and \( 3Q + 7R \) using a CAS/calculator.

To calculate \( 4Q - 2R \) by hand, the steps are:

\[
4Q - 2R = 4 \begin{bmatrix} 3 & -5 & 10 \\ 11 & 6 & 17 \\ -2 & 15 & 0 \end{bmatrix} - 2 \begin{bmatrix} 12 & 2 & 1 \\ -3 & 4 & 7 \\ 22 & 14 & -7 \end{bmatrix} = \begin{bmatrix} 12 & -20 & 40 \\ 44 & 24 & 68 \\ -8 & 60 & 0 \end{bmatrix} - \begin{bmatrix} 24 & 4 & 2 \\ -6 & 8 & 14 \\ 44 & 28 & -14 \end{bmatrix} = \begin{bmatrix} -12 & -24 & 38 \\ 50 & 16 & 54 \\ -52 & 32 & 14 \end{bmatrix}
\]

Using a CAS/calculator is much quicker, particularly if there are repeated calculations using the same matrices, where it is useful to store the matrices first.

TI-Nspire CAS

**STEP 1**

Use a new document with a calculator page. 
Press \( \text{ctrl} \) [VAR], then select the 3 \( \times \) 3 matrix template. 
Set the number of rows to 3 and the number of columns to 3.

Create the first 3 \( \times \) 3 matrix, then press \( \text{ctrl} \) [VAR] \( Q \) [ENTER] to store this matrix as \( q \).

Create the second 3 \( \times \) 3 matrix, then press \( \text{ctrl} \) [VAR] \( R \) to store this matrix as \( r \).

**STEP 2**

The necessary calculations can then be performed.

Open a new document to clear the stored matrices.
**CLASSPAD**

**STEP 1**

Use the application.

Press \( \text{Main} \), then select \( \text{Math2} \) and tap the 2 \( \times \) 2 matrix template twice for a 3 \( \times \) 3 matrix. Create the first 3 \( \times \) 3 matrix, then tap \( \text{Var} \), \( \text{MATS} \) Q and press \( \text{EXE} \) to store this matrix as Q.

Create the second 3 \( \times \) 3 matrix, then tap \( \text{Var} \) R and press \( \text{EXE} \) to store this matrix as R.

**STEP 2**

The necessary calculations can then be performed.

To clear the stored matrices, tap \( \text{Edit} \), then \( \text{Clear All variables} \), then \( \text{OK} \).
Find the values of the pronumerals in the following matrix equation.

\[
3 \begin{bmatrix} x & 2 & -1 \\ 4 & 5 & y \end{bmatrix} + 2 \begin{bmatrix} 14 & 16 & 11 \end{bmatrix} = \begin{bmatrix} 14 & 16 & 11 \end{bmatrix}
\]

**Working**

1. Perform the scalar multiplications on the left-hand side.

\[
3 \begin{bmatrix} x & 2 & -1 \\ 4 & 5 & y \end{bmatrix} + 2 \begin{bmatrix} 14 & 16 & 11 \end{bmatrix} = \begin{bmatrix} 3x & 6 & -3 \\ 8 & 10 & 2y \end{bmatrix}
\]

2. Add corresponding elements

\[
\begin{bmatrix} 3x & 6 & -3 \\ 8 & 10 & 2y \end{bmatrix} = \begin{bmatrix} 3x + 8 & 16 & -3 + 2y \\ 14 & 16 & 11 \end{bmatrix}
\]

3. Now equate it to the right-hand side of the original equation.

\[
\begin{bmatrix} 3x + 8 & 16 & -3 + 2y \end{bmatrix} = \begin{bmatrix} 14 & 16 & 11 \end{bmatrix}
\]

4. Equate corresponding elements and solve for the unknown pronumerals.

\[
x = 2 \\
y = 7
\]

**Addition, subtraction and scalar multiplication of matrices**

**Prep 1**  
**WORKED EXAMPLE 3**

Using the matrices \(N\), \(O\), \(P\) and \(Q\), find the following.

\[
N = \begin{bmatrix} 16 & 5 \\ 2 & 12 \end{bmatrix}, \quad O = \begin{bmatrix} 10 & 3 \\ 7 & 22 \end{bmatrix}, \quad P = \begin{bmatrix} 15 & 1 \\ 8 & 34 \end{bmatrix}, \quad Q = \begin{bmatrix} 11 & -5 \\ 29 & 36 \end{bmatrix}
\]

\[
a \quad N + O \\
b \quad N + P \\
c \quad O + P \\
d \quad Q - N \\
e \quad N - Q \\
f \quad P - P
\]

**Prep 2**  
**USING CAS: ADDING AND SUBTRACTING MATRICES**

Use a CAS/calculator to evaluate the following.

\[
a \begin{bmatrix} 0 & 5 & -1.2 \\ 3.6 & -2 & 10 \\ 14 & 5.1 & -6 \end{bmatrix} + \begin{bmatrix} 3.1 & 8 & 6.2 \\ 5 & 12 & -16 \\ 3 & -3.2 & -5 \end{bmatrix}
\]

\[
b \begin{bmatrix} 8 & 5.2 & 17 \\ 5.8 & 12 & 16 \\ 18 & 9.2 & -11 \end{bmatrix} - \begin{bmatrix} 0 & 3.1 & 9 \\ 5 & 12 & -16 \\ 3 & -3.2 & -5 \end{bmatrix}
\]

16  Nelson VCE General Mathematics Unit 2
Prep 3  WORKED EXAMPLE 4

Find the value of the pronumeral in each of the following.

\[
\begin{bmatrix}
21 \\
35 \\
16 \\
15
\end{bmatrix} + \begin{bmatrix}
17 \\
9 \\
38 \\
14
\end{bmatrix} = \begin{bmatrix}
38 \\
m \\
54 \\
29
\end{bmatrix}
\]

\[
\begin{bmatrix}
27 \\
13 \\
9 \\
44
\end{bmatrix} - \begin{bmatrix}
11 \\
14 \\
28 \\
39
\end{bmatrix} = \begin{bmatrix}
16 \\
-1 \\
-19 \\
d
\end{bmatrix}
\]

\[
\begin{bmatrix}
12 \\
3 \\
16 \\
4 \\
9 \\
27
\end{bmatrix} + \begin{bmatrix}
17 \\
x \\
31 \\
2 \\
13 \\
40
\end{bmatrix} = \begin{bmatrix}
29 \\
28 \\
47 \\
6 \\
22 \\
67
\end{bmatrix}
\]

\[
\begin{bmatrix}
31 \\
19 \\
28 \\
t
\end{bmatrix} + \begin{bmatrix}
56 \\
47 \\
37 \\
29
\end{bmatrix} = \begin{bmatrix}
87 \\
66 \\
65 \\
77
\end{bmatrix}
\]

\[
\begin{bmatrix}
51 \\
36 \\
48
\end{bmatrix} - \begin{bmatrix}
15 \\
m \\
60
\end{bmatrix} = \begin{bmatrix}
36 \\
7 \\
-12
\end{bmatrix}
\]

\[
\begin{bmatrix}
14 \\
3 \\
27 \\
9 \\
11 \\
-5
\end{bmatrix} - \begin{bmatrix}
-12 \\
6 \\
13 \\
-10 \\
14 \\
n
\end{bmatrix} = \begin{bmatrix}
2 \\
-3 \\
14 \\
19 \\
-3 \\
-15
\end{bmatrix}
\]

Prep 4  USING CAS: ADDITION, SUBTRACTION AND SCALAR MULTIPLICATION OF MATRICES

Given that \( M = \begin{bmatrix} 5 & 9 \\ 3 & 8 \end{bmatrix} \) and \( N = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix} \), evaluate the following using a CAS/calculator.

\( a \) \( 2M + 2N \)  \( b \) \( 3N - 2M \)  \( c \) \( 5M + 4N \)

\( d \) \( 6M - 4N \)  \( e \) \( 12N - 3M \)  \( f \) \( M + 2M \)

Prep 5  WORKED EXAMPLE 5

Find the values of the pronumerals in the following matrix equations.

\[
\begin{bmatrix}
a \\
3 \\
7 \\
-2
\end{bmatrix} + 2 \begin{bmatrix}
5 \\
b \\
11 \\
8
\end{bmatrix} = \begin{bmatrix}
25 \\
37 \\
57 \\
c
\end{bmatrix}
\]

\[
\begin{bmatrix}
10 \\
5 \\
4 \\
x \\
8 \\
7
\end{bmatrix} + 4 \begin{bmatrix}
1 \\
9 \\
8 \\
y \\
6 \\
2
\end{bmatrix} = \begin{bmatrix}
34 \\
51 \\
z \\
45 \\
60 \\
29
\end{bmatrix}
\]

\[
\begin{bmatrix}
5 \\
p \\
7 \\
9
\end{bmatrix} - 2 \begin{bmatrix}
11 \\
10 \\
q \\
8
\end{bmatrix} = \begin{bmatrix}
2f \\
32 \\
-10 \\
20
\end{bmatrix}
\]

\[
\begin{bmatrix}
a \\
-3 \\
-7 \\
-10
\end{bmatrix} - \begin{bmatrix}
12 \\
c \\
6 \\
d
\end{bmatrix} = \begin{bmatrix}
-52 \\
23 \\
19 \\
d
\end{bmatrix}
\]
Addition, subtraction and scalar multiplication of matrices

**Question 1**

\[
\begin{bmatrix}
27 & 32 & 8 & 17
\end{bmatrix}
- \begin{bmatrix}
18 & 12 & 20 & 5
\end{bmatrix}\]

is equal to

A \[29\]  
B \[9 \ 20 \ -12 \ 12\]  
C \[9 \ 20 \ -12 \ 12\]  
D \[\begin{bmatrix}
9 \\
20 \\
-12 \\
12
\end{bmatrix}\]  
E \[\begin{bmatrix}
27 & 32 & 8 & 17
\end{bmatrix}
- \begin{bmatrix}
18 & 12 & 20 & 5
\end{bmatrix}\]

**Question 2**

If \(C = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}\), then \(5C\) equals

A \[5\]  
B \[\begin{bmatrix}
5 & 0 \\
0 & 5
\end{bmatrix}\]  
C \[\begin{bmatrix}
6 & 5 \\
5 & 6
\end{bmatrix}\]  
D \[\begin{bmatrix}
0 & 5 \\
5 & 0
\end{bmatrix}\]  
E \[\begin{bmatrix}
5 & 6 \\
6 & 5
\end{bmatrix}\]

**Question 3**

\[2 \begin{bmatrix}
3 & 2 \\
0 & 4
\end{bmatrix} + 3 \begin{bmatrix}
-1 & 0 \\
1 & 6
\end{bmatrix}\]

is equal to

A \[\begin{bmatrix}
5 & 2 \\
2 & 10
\end{bmatrix}\]  
B \[\begin{bmatrix}
3 & 7 \\
3 & 26
\end{bmatrix}\]  
C \[\begin{bmatrix}
6 & 2 \\
2 & 10
\end{bmatrix}\]  
D \[\begin{bmatrix}
3 & 4 \\
3 & 26
\end{bmatrix}\]  
E \[\begin{bmatrix}
5 & 4 \\
3 & 8
\end{bmatrix}\]

**Question 4**

The matrix sum \[\begin{bmatrix}
0 & -4 \\
2 & 5
\end{bmatrix} + \begin{bmatrix}
5 & 4 \\
-2 & 2
\end{bmatrix}\]

is equal to

A \[\begin{bmatrix}
5 & 0 \\
0 & 7
\end{bmatrix}\]  
B \[\begin{bmatrix}
0 & 0 \\
0 & 7
\end{bmatrix}\]  
C \[\begin{bmatrix}
5 & -4 \\
0 & 7
\end{bmatrix}\]  
D \[\begin{bmatrix}
0 & 5 & -4 & 4 \\
2 & -2 & 5 & 2
\end{bmatrix}\]  
E \[\begin{bmatrix}
0 & -4 & 5 & 4 \\
2 & 5 & -2 & 2
\end{bmatrix}\]

[VCAA 2007 1MQ1]
### Question 5

$$2 \times \begin{bmatrix} 2 & 8 \\ 4 & -1 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 7 \\ 4 & 2 \\ 2 & 3 \end{bmatrix}$$ equals

A

$$\begin{bmatrix} 1 & 1 \\ 0 & -3 \\ 4 & 2 \end{bmatrix}$$

B

$$\begin{bmatrix} -2 & 2 \\ 0 & -6 \\ 2 & 4 \end{bmatrix}$$

C

$$\begin{bmatrix} 1 & 9 \\ 12 & 0 \\ 8 & 13 \end{bmatrix}$$

D

$$\begin{bmatrix} 1 & 9 \\ 4 & -4 \\ 4 & 7 \end{bmatrix}$$

E

$$\begin{bmatrix} -1 & 1 \\ 0 & -3 \\ 1 & 2 \end{bmatrix}$$

[VCAA 2012 1MQ1]

### Question 6

If

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 7 \\ 8 & d \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 8 & 11 \end{bmatrix}$$

then $d$ is equal to

A

$$-11$$

B

$$-10$$

C

$$7$$

D

$$10$$

E

$$11$$

[VCAA 2008 1MQ1]

### Question 7

$$3 \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} + 2 \begin{bmatrix} -1 & 0 \\ 2 & -7 \end{bmatrix}$$ equals

A

$$\begin{bmatrix} 4 & 3 \\ 4 & -5 \end{bmatrix}$$

B

$$\begin{bmatrix} 6 & 1 \\ 1 & -4 \end{bmatrix}$$

C

$$\begin{bmatrix} 4 & 3 \\ 4 & 2 \end{bmatrix}$$

D

$$\begin{bmatrix} 5 & 1 \\ 1 & -4 \end{bmatrix}$$

E

$$\begin{bmatrix} 3 & 6 \\ 7 & 4 \end{bmatrix}$$

[VCAA 2009 1MQ1]

### Question 8

The matrix $$\begin{bmatrix} 12 & 36 \\ 0 & 24 \end{bmatrix}$$ is equal to

A

$$12 \begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix}$$

B

$$12 \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

C

$$12 \begin{bmatrix} 0 & 24 \\ -12 & 12 \end{bmatrix}$$

D

$$12 \begin{bmatrix} 0 & 24 \\ 0 & 12 \end{bmatrix}$$

E

$$12 \begin{bmatrix} 1 & 3 \\ -12 & 2 \end{bmatrix}$$

[VCAA 2006 1MQ1]

### Question 9

Evaluate $5R - 3T$ if $R = \begin{bmatrix} 7 & 3 \\ 6 & 8 \end{bmatrix}$ and $T = \begin{bmatrix} 10 & 6 \\ 7 & 9 \end{bmatrix}$.

$$\begin{bmatrix} -3 & -3 \\ -1 & -1 \end{bmatrix}$$

B

$$\begin{bmatrix} 35 & 30 & 5 \\ 30 & 21 & 9 \\ 15 & 18 & -3 \\ 40 & 27 & 13 \end{bmatrix}$$

C

$$\begin{bmatrix} 5 & -3 \\ 9 & 13 \end{bmatrix}$$

D

$$\begin{bmatrix} -29 & -21 \\ -17 & -21 \end{bmatrix}$$

E

$$\begin{bmatrix} -1 & -3 \\ -3 & -1 \end{bmatrix}$$
Matrix multiplication

We have looked at the multiplication of a matrix by a scalar. It is also possible to multiply a matrix by a matrix. To multiply two matrices, we multiply pairs of elements, working across the rows in the first matrix and down the columns in the second matrix.

For example, if we have

\[ A = \begin{bmatrix} 2 & 6 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 5 & 1 \\ 2 & 4 & 7 \end{bmatrix}, \]

and we need to find \( AB \), the first element in \( AB \) is calculated by multiplying each element of \( A \) row 1 by its matching element in \( B \) column 1 and adding them together.

\[ AB = \begin{bmatrix} 2 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 & 1 \\ 2 & 4 & 7 \end{bmatrix}, \]

so the first element of \( AB \) is \((2 \times 3 + 6 \times 2) = 18\).

Follow this pattern to find all the elements of \( AB \).

\[
\begin{bmatrix}
(2 \times 3) + (6 \times 2) = 18 & (2 \times 5) + (6 \times 4) = 34 & (2 \times 1) + (6 \times 7) = 44 \\
(0 \times 3) + (1 \times 2) = 2 & (0 \times 5) + (1 \times 4) = 4 & (0 \times 1) + (1 \times 7) = 7
\end{bmatrix}
\]

So \( AB = \begin{bmatrix} 18 & 34 & 44 \\ 2 & 4 & 7 \end{bmatrix} \)

**Exam hack**

Note that the row and column pairs you multiply together also tell you the position of the element in the \( AB \) matrix. For example, when you multiply 'A row 1 \( \times \) B column 3', the result is the \( AB \) element in row 1 column 3.
Let \( A = \begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix} \).

a  Calculate \( AB \) by hand and verify your answer using a CAS/calculator.

b  Calculate \( BA \) by hand and verify your answer using a CAS/calculator.

c  What do you notice about the two products?

To solve \( AB \) and \( BA \) by hand, the steps are:

\[
AB = \begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 5 \times 1 + 3 \times 3 & 5 \times 5 + 3 \times 7 \\ 2 \times 1 + 6 \times 3 & 2 \times 5 + 6 \times 7 \end{bmatrix} = \begin{bmatrix} 14 & 46 \\ 20 & 52 \end{bmatrix}
\]

\[
BA = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 \times 5 + 5 \times 2 & 1 \times 3 + 5 \times 6 \\ 3 \times 5 + 7 \times 2 & 3 \times 3 + 7 \times 6 \end{bmatrix} = \begin{bmatrix} 15 & 33 \\ 29 & 51 \end{bmatrix}
\]

To calculate using a CAS/calculator, enter the matrices and assign them as in the previous Using CAS on pages 14-15 before doing the multiplication.

**c**  \( AB \) and \( BA \) are not equal. Swapping the order when multiplying matrices has resulted in different answers.

Unlike when multiplying numbers, swapping the order when multiplying matrices usually gives a different answer. Generally for matrices \( AB \neq BA \).

Since the order in which you multiply matrices is important, we have two terms to describe different sorts of multiplication. When we are multiplying two matrices \( AB \), we say we are **pre-multiplying** by \( A \) (i.e. \( A \) is before \( B \)) or **post-multiplying** by \( B \) (i.e. \( B \) is after \( A \)).
Matrix multiplication rules

Multiplication between two matrices is only possible if the number of columns in the first matrix is the same as the number of rows in the second matrix. Furthermore, the product matrix will have the same number of rows as the first matrix and the same number of columns as the second matrix.

### Matrix product rule

If matrix $A$ is of order $m \times n$ and matrix $B$ is of order $n \times p$, then the product $AB$ is defined, which means it’s possible. Otherwise $AB$ isn’t defined, which means it’s not possible.

- **Order of 1st matrix**: $(m \times n)$
- **Order of 2nd matrix**: $(n \times p)$

These tell us the order of $AB$.

For example, $(3 \times 7) \times (7 \times 6)$ is a defined product, but $(3 \times 2) \times (3 \times 2)$ isn’t.

### Order of product matrix rule

If matrix $A$ is of order $m \times n$ and matrix $B$ is of order $n \times p$, then $AB$ will be of order $m \times p$.

This can be shown by a **matrix order equation**:

$$\begin{align*}
\text{order of 1st matrix} &\quad \times \quad \text{order of 2nd matrix} \\
(m \times n) &\quad \times \quad (n \times p) \\
\text{order of product matrix} &\quad = \quad (m \times p)
\end{align*}$$

For example, the matrix order equation $(3 \times 4) \times (4 \times 6) = (3 \times 6)$ tells us that a matrix with 3 rows and 4 columns multiplied by a matrix with 4 rows and 6 columns will result in a matrix with 3 rows and 6 columns.

### Exam hack

The matrix order equation can be used to work out the order of a product when more than two matrices are being multiplied together. For example, $(3 \times 7) \times (7 \times 2) \times (2 \times 5) = (3 \times 5)$.

This means, if the multiplications are defined:
- the number of rows in the product = the number rows in the first matrix
- the number of columns in the product = the number of columns in the last matrix
Summing matrices

Row and column matrices that consist entirely of 1s are called summing matrices because when they are multiplied with other matrices, they sum elements. For example,

\[
\begin{bmatrix}
1 & 1 \\
3 & 7 & 5
\end{bmatrix}
\begin{bmatrix}
10 & 2 & 1 \\
2 & 7 & 1
\end{bmatrix} =
\begin{bmatrix}
13 & 9 & 6 \\
2 & 7 & 1
\end{bmatrix}
\]

where the columns are summed.

\[
\begin{bmatrix}
2 & 4 & 3 \\
5 & 1 & 9 & 6
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix} =
\begin{bmatrix}
9 & 15 \\
13 & 27
\end{bmatrix}
\]

where the rows are summed.

To sum the columns of a matrix, pre-multiply it by a suitable row matrix consisting of 1s.
To sum the rows of a matrix, post-multiply it by a suitable column matrix consisting of 1s.

### Worked example 6

Let \( A = \begin{bmatrix} 3 & 7 & 2 & 12 \end{bmatrix}, \ B = \begin{bmatrix} 5 \\ 9 \\ 7 \\ 2 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \ D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ X = \begin{bmatrix} 3 & 2 \\ 4 & 6 \\ 5 & 10 \end{bmatrix}, \ Y = \begin{bmatrix} 18 & 2 \\ 12 & 5 \end{bmatrix}. \)

**a i** Show how the orders of \( A \) and \( B \) will tell us what the order of \( AB \) will be.  
**ii*** Calculate \( AB \).

**b i** Show how the orders of \( A \) and \( B \) will tell us what the order of \( BA \) will be.  
**ii*** Calculate \( BA \).

**c** Calculate \( CX \) and comment on what has been summed.

**d** Calculate \( YD \) and comment on what has been summed.

**Working**

**a i** Use the fact that \((m \times n) \times (n \times p)\) gives an \(m \times p\) matrix.

\( A \) is of order \(1 \times 4\)
\( B \) is of order \(4 \times 1\)
\( AB \) is \((1 \times 4) \times (4 \times 1)\)
So \( AB \) will be a \(1 \times 1\) matrix.

**ii*** Multiply the corresponding elements, then add them together.

\[
AB = \begin{bmatrix} 3 & 7 & 2 & 12 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \\ 7 \\ 2 \end{bmatrix}
= \begin{bmatrix} 3 \times 5 + 7 \times 9 + 2 \times 7 + 12 \times 2 \end{bmatrix}
= \begin{bmatrix} 116 \end{bmatrix}
\]
b i Use the fact that \((m \times n) \times (n \times p)\) gives an \(m \times p\) matrix.

\[ B \text{ is of order 4} \times 1 \]
\[ A \text{ is of order 1} \times 4 \]
\[ BA \text{ is (4} \times 1) \times (1 \times 4) \]
So \(BA\) will be a 4 \(\times\) 4 matrix.

ii Multiply the corresponding elements, then add them together.

\[ BA = \begin{bmatrix} 5 & 9 & 3 & 7 & 2 & 12 \\ 7 & 2 \\ \\ 15 & 35 & 10 & 60 \\ 27 & 63 & 18 & 108 \\ 21 & 49 & 14 & 84 \\ 6 & 14 & 4 & 24 \end{bmatrix} \]

\[ = \begin{bmatrix} 5 & 7 \\ 15 & 35 & 10 & 60 \\ 27 & 63 & 18 & 108 \\ 21 & 49 & 14 & 84 \\ 6 & 14 & 4 & 24 \end{bmatrix} \]

\[ X \text{ is of order 3} \times 2, \text{ } A \text{ is of order 2} \times 2 \]
\[ \text{and } B \text{ is of order 3} \times 2. \]

\[ X \text{ has order } 3 \times 2. \]
\[ A \text{ has order } 2 \times 2. \]
\[ B \text{ has order } 3 \times 2. \]

\[ XA \text{ is } (3 \times 2) \times (2 \times 2) \]
\[ XB \text{ is } (3 \times 2) \times (3 \times 2) \]
\[ AX \text{ is } (2 \times 2) \times (3 \times 2). \]

\[ XA \text{ is defined. } \]
\[ XB \text{ is not defined. } \]
\[ AX \text{ is not defined. } \]

\[ XA \text{ has order } 3 \times 2. \]
\[ XB \text{ does not exist. } \]
\[ AX \text{ does not exist. } \]

\[ YD = \begin{bmatrix} 18 & 2 & 1 \\ 12 & 5 & 1 \end{bmatrix} \]

The columns of \(D\) have been summed.

\[ YD = \begin{bmatrix} 20 \\ 17 \end{bmatrix} \]

The rows of \(D\) have been summed.
Powers of matrices

Just as \(5^2\) means \(5 \times 5\), if we have a matrix named \(A\), then \(A^2\) means \(A \times A\).

Only square matrices can be raised to a power because otherwise the number of columns in the first matrix will be different to the number of rows in the second matrix and multiplication won’t be possible. For example, a \(2 \times 3\) matrix squared would give \((2 \times 3) \times (2 \times 3)\), which isn’t defined.

Only square matrices can be raised to a power.

Any power of a matrix will always have the same order as the original matrix.

Using CAS Finding powers of matrices

If \(X = \begin{bmatrix} 5 & 11 & 12 \\ 7 & 3 & 1 \\ -4 & 2 & 6 \end{bmatrix}\),

\(a\) find \(X^4\).

\(b\) What order do you think \(X^{50}\) will have?

\(a\) Create a \(3 \times 3\) matrix and enter the elements as shown.

\(b\) The order of \(X^{50}\) will be \(3 \times 3\), the same as the order of \(X\).
Multiplying matrices

 Prep 1  USING CAS: MULTIPLYING MATRICES

Let \( X = \begin{bmatrix} 5 & 8 & 11 \\ 3 & 6 & 4 \end{bmatrix} \) and \( Y = \begin{bmatrix} 1 & 5 \\ 6 & 4 \\ 3 & 7 \end{bmatrix} \).

a  Calculate \( XY \) by hand and verify your answer using a CAS/calculator.
b  Calculate \( YX \) by hand and verify your answer using a CAS/calculator.
c  What do you notice about the two products?

 Prep 2  WORKED EXAMPLE 6

Let \( A = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 10 \end{bmatrix}, B = \begin{bmatrix} 3 \\ 4 \\ 10 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \\ 5 \\ 2 \end{bmatrix} \) and \( X = \begin{bmatrix} 6 & 4 \\ 1 & 3 \\ 5 & 2 \end{bmatrix} \) and \( Y = \begin{bmatrix} 9 & 20 \\ 3 & 9 \end{bmatrix} \).

a  i  Show how the orders of \( A \) and \( B \) will tell us what the order of \( AB \) will be.
   ii  Calculate \( AB \).
b  i  Show how the orders of \( A \) and \( B \) will tell us what the order of \( BA \) will be.
   ii  Calculate \( BA \).
c  Calculate \( CX \) and comment on what has been summed.
d  Calculate \( YD \) and comment on what has been summed.

 Prep 3  WORKED EXAMPLE 7

Let \( A = \begin{bmatrix} 2 & 4 & 1 \\ 5 & 6 & 3 \end{bmatrix}, B = \begin{bmatrix} 5 & 4 \\ 6 & 8 \end{bmatrix}, C = \begin{bmatrix} 2 & 3 \\ 5 & 2 \\ 1 & 6 \\ 4 & 7 \end{bmatrix} \) and \( D = \begin{bmatrix} 4 & 3 \\ 5 & 1 \\ 2 & 6 \end{bmatrix} \).

For each matrix product below:
   i  state whether the product is defined
   ii  for those products that are defined, state their order.

\[- a  AB \quad b  BA \quad c  AD \quad d  AC \quad e  DA \]
\[- f  BC \quad g  CB \quad h  DB \quad i  CD \quad j  BB \]
If \( X = \begin{bmatrix} 3 & 7 \\ 9 & 4 \end{bmatrix} \) and \( Y = \begin{bmatrix} 2.5 & 6.1 \\ 3.9 & 2.7 \end{bmatrix} \), find

\[ a \ X^2 \quad b \ X^3 \quad c \ Y^2 \quad d \ X^2 + Y^2 \]
\[ e \ X^5 \quad f \ X^5 - X^2 \quad g \ \text{what order} \ X^{41} - Y^{26} \ \text{has} \]

**EXAM PRACTICE 7.3**

**Multiplying matrices**

**Question 1**

If the matrix product \( AB \) exists, where 
\[ A = \begin{bmatrix} 6 & 12 \\ 8 & 19 \\ 21 & 4 \\ 17 & 32 \\ 5 & 29 \end{bmatrix} \]

then which one of the following could be the matrix \( B \)?

\[ A \begin{bmatrix} 5 & 6 & 3 & 1 \end{bmatrix} \]
\[ B \begin{bmatrix} 2 & 2 \\ 9 & 11 \\ 1 & 1 \end{bmatrix} \]
\[ C \begin{bmatrix} 3 & 2 \\ 4 & 9 \\ 7 & 4 \end{bmatrix} \]
\[ D \begin{bmatrix} 2 & 2 \\ 13 & 4 \end{bmatrix} \]
\[ E \begin{bmatrix} 4 & 11 \\ 1 & 1 \end{bmatrix} \]

**Question 2**

If 
\[ P = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \]
\[ Q = \begin{bmatrix} 2 & -3 & 4 \end{bmatrix} \]

then the order of \( PQ \) is

\[ A \ 1 \times 1 \quad B \ 3 \times 1 \quad C \ 1 \times 3 \quad D \ 3 \times 3 \quad E \ 2 \times 3 \]

**Question 3**

Which one of the following products cannot be found?

\[ A \begin{bmatrix} 2 & 7 & 5 \\ 6 & 2 \\ 3 \end{bmatrix} \]
\[ B \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 20 & 12 \end{bmatrix} \]
\[ C \begin{bmatrix} 6 & 1 \\ 7 & 2 \\ 5 \end{bmatrix} \]
\[ D \begin{bmatrix} 7 & 15 \\ 3 & 6 \\ 72 & 12 \\ 11 & 1 \end{bmatrix} \]
\[ E \begin{bmatrix} 0.5 & 16 \\ 3.8 & 2 \\ 14 & 2.1 \end{bmatrix} \]
Question 4

If \( G = \begin{bmatrix}
5 & 4 & 9 \\
2 & 3 & 7 \\
6 & 8 & 5 \\
0 & 1 & 5
\end{bmatrix} \) and \( H = \begin{bmatrix}
7 & 2 \\
5 & 8 \\
1 & 9
\end{bmatrix} \), then \( GH \) will be of order

A \( 4 \times 2 \)      B \( 12 \times 6 \)      C \( 4 \times 3 \)      D \( 2 \times 4 \)      E \( 6 \times 12 \)

Question 5

If \( A = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \), \( B = \begin{bmatrix}
1 \\
0
\end{bmatrix} \) and \( C = \begin{bmatrix}
0 \\
1
\end{bmatrix} \), then \( AB + 2C \) equals

A \( \begin{bmatrix}
0 \\
3
\end{bmatrix} \)      B \( \begin{bmatrix}
3 \\
0
\end{bmatrix} \)      C \( \begin{bmatrix}
1 \\
2
\end{bmatrix} \)      D \( \begin{bmatrix}
2 \\
0
\end{bmatrix} \)      E \( \begin{bmatrix}
2 \\
3
\end{bmatrix} \)

[VCAA 2011 1MQ2]

Question 6

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
2 \\
0 \\
-2 \\
0
\end{bmatrix}
= \begin{bmatrix}
0 \\
-1 \\
-1 \\
0
\end{bmatrix}
\]

equals

A \( \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} \)      B \( \begin{bmatrix}
2 \\
2 \\
2 \\
2
\end{bmatrix} \)      C \( \begin{bmatrix}
2 \\
2 \\
-2 \\
2
\end{bmatrix} \)      D \( \begin{bmatrix}
4 \\
0 \\
0 \\
4
\end{bmatrix} \)      E \( \begin{bmatrix}
0 \\
2 \\
2 \\
0
\end{bmatrix} \)

[VCAA 2013 1MQ1]

Question 7

The matrix \( \begin{bmatrix}
12 & 15 & 3 \\
-6 & 0 & 24
\end{bmatrix} \) can also be written as

A \( \begin{bmatrix}
12 & 15 & 3
\end{bmatrix} + \begin{bmatrix}
-6 & 0 & 24
\end{bmatrix} \)

B \( \begin{bmatrix}
12 \\
-6
\end{bmatrix} + \begin{bmatrix}
15 \\
0
\end{bmatrix} + \begin{bmatrix}
3 \\
24
\end{bmatrix} \)

C \( \begin{bmatrix}
3 \\
6
\end{bmatrix} + \begin{bmatrix}
4 & 5 & 1 \\
-1 & 0 & 4
\end{bmatrix} \)

D \( \frac{1}{3} \begin{bmatrix}
4 & 5 & 1 \\
-2 & 0 & 8
\end{bmatrix} \)

E \( 3 \times \begin{bmatrix}
4 & 5 & 1 \\
-2 & 0 & 8
\end{bmatrix} \)

[VCAA 2009 1MQ2]
**Question 8**

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

Then $A^3(B - C) =$

\[
\begin{align*}
A &= \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} & B &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & C &= \begin{bmatrix} 3 & 6 \\ 6 & -3 \end{bmatrix} \\
D &= \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} & E &= \begin{bmatrix} 5 & 10 \\ 10 & -5 \end{bmatrix}
\end{align*}
\]

[VCAA 2006 1MQ3]

**Question 9**

Matrix $A$ has three rows and two columns. Matrix $B$ has four rows and three columns. Matrix $C = B \times A$ has

A two rows and three columns.  
B three rows and two columns.  
C three rows and three columns.  
D four rows and two columns.  
E four rows and three columns.

[VCAA 2013 1MQ2]

**Question 10**

Let $A = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 9 \end{bmatrix}$ and $C = \begin{bmatrix} 2 \end{bmatrix}$

Using these matrices, the matrix product that is *not* defined is

A $AB$  
B $AC$  
C $BA$  
D $BC$  
E $CB$

[VCAA 2006 1MQ2]

**Question 11**

If $A = \begin{bmatrix} 8 & 1 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 12 \\ 6 & 0 \end{bmatrix}$, then matrix $AB = \begin{bmatrix} 30 & 96 \\ 24 & 48 \end{bmatrix}$.

The element '24' in the matrix $AB$ is correctly obtained by calculating

A $4 \times 6 + 2 \times 0$  
B $4 \times 3 + 2 \times 6$  
C $3 \times 4 + 12 \times 1$  
D $4 \times 2 + 8 \times 2$  
E $8 \times 3 + 1 \times 0$

[VCAA 2012 1MQ2]
Question 12

Which matrix expression results in a matrix that contains the sum of the numbers 2, 5, 4, 1 and 8?

A \[ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 & 5 & 4 & 1 & 8 \end{bmatrix} \]

B \[ \begin{bmatrix} 2 & 5 & 4 & 1 & 8 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \]

C \[ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix} \]

D \[ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 5 \\ 4 \\ 1 \\ 8 \end{bmatrix} \]

E \[ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix} \]

[VCAA 2010 1MQ4]

Question 13

Given that \[ A = \begin{bmatrix} 5 & 4 \\ 11 & 8 \\ 7 & 6 \end{bmatrix} \], \[ B = \begin{bmatrix} 2 & 9 & 12 \\ 10 & 5 & 6 \end{bmatrix} \], \[ C = \begin{bmatrix} 3 & 8 \\ 1 & 0 \end{bmatrix} \] and \[ D = \begin{bmatrix} 1 & 7 \\ 3 & 8 \\ 9 & 2 \end{bmatrix} \],

answer the following questions.

a Which pair of matrices can be added?

b Does the matrix product \( AC \) exist? Explain.

c List all of the possible matrix products formed by pairs from the given matrices.

d State the order of each matrix product that you listed for part c.

e Which matrix can be raised to a power? Why?
The identity matrix

If we multiply a number by 1, it leaves the original number the same. The number 1 is called the multiplicative identity because the original number remains identical to what it was before multiplication.

The equivalent matrix to the number 1 is the identity matrix, \( I \), the square matrix where all the elements in the leading diagonal are 1 and the other elements are 0. Multiplying a matrix by the identity matrix leaves the original matrix unchanged. Here are some identity matrices:

\[
I = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix},
I = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
I = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Worked example 8

If \( A = \begin{bmatrix}
3 & 10 \\
2 & 5
\end{bmatrix} \) and \( I \) is the identity matrix, find \( AI \) and \( IA \) by hand.

What do you notice when you compare the two answers?

**Working**

1. Calculate \( AI \).
   \[
   AI = \begin{bmatrix}
3 & 10 \\
2 & 5
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
3\times1+10\times0 & 3\times0+10\times1 \\
2\times1+5\times0 & 2\times0+5\times1
\end{bmatrix} = \begin{bmatrix}
3 & 10 \\
2 & 5
\end{bmatrix}
   \]

2. Calculate \( IA \).
   \[
   AI = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
3 & 10 \\
2 & 5
\end{bmatrix} = \begin{bmatrix}
1\times3+0\times2 & 1\times10+0\times5 \\
0\times3+1\times2 & 0\times10+1\times5
\end{bmatrix} = \begin{bmatrix}
3 & 10 \\
2 & 5
\end{bmatrix}
   \]

3. Compare the two answers.
   \( AI = IA = A \)

The identity matrix, \( I \), is the square matrix with leading diagonal elements 1 and other elements 0, which when multiplied by a square matrix, \( A \), of the same order, leaves \( A \) unchanged.

\( AI = IA = A \)
The inverse matrix

The multiplicative inverse of the number 5 is \(\frac{1}{5}\). It is what you multiply the original number by to get 1. For example, \(5 \times \frac{1}{5} = \frac{5}{5} = 1\). Similarly, the multiplicative inverse of a matrix is what you multiply the original matrix by to get the identity matrix, \(I\).

Show that the matrices
\[
\begin{pmatrix}
3 & 7 \\
2 & 5
\end{pmatrix}
\text{ and }
\begin{pmatrix}
5 & -7 \\
-2 & 3
\end{pmatrix}
\]
are inverses of each other.

**Working**

1. Show that the product of the two matrices equals the identity matrix.

\[
\begin{pmatrix}
3 & 7 \\
2 & 5
\end{pmatrix}
\times
\begin{pmatrix}
5 & -7 \\
-2 & 3
\end{pmatrix}
= \begin{pmatrix}
15 - 14 & -21 + 21 \\
10 - 10 & -14 + 15
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

2. Reverse the order of the two matrices and show that their product also equals the identity matrix.

\[
\begin{pmatrix}
5 & -7 \\
-2 & 3
\end{pmatrix}
\times
\begin{pmatrix}
3 & 7 \\
2 & 5
\end{pmatrix}
= \begin{pmatrix}
15 - 14 & 35 - 35 \\
-6 + 6 & -14 + 15
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]
Finding the determinant and the inverse of a matrix

To find the inverse of a matrix, we first need to find the determinant. We will start by looking at $2 \times 2$ matrices. The calculations involved for finding determinants and inverses of higher order square matrices are significantly more complex and require a CAS/calculator.

For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $\det(A) = ad - bc$ and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The inverse doesn't exist when $\det(A) = 0$.

Worked example 10

For each of the following matrices find

i. the determinant

ii. the inverse (if it exists).

a. $A = \begin{bmatrix} 6 & 2 \\ 8 & 3 \end{bmatrix}$

b. $B = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$

c. $C = \begin{bmatrix} 8 & 2 \\ 8 & 2 \end{bmatrix}$

**Working**

a. i. Calculate $\det(A)$.

$$\det(A) = 6 \times 3 - 2 \times 8 = 18 - 16 = 2$$

ii. Find $A^{-1}$ (if it exists).

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 3 & -2 \\ -8 & 6 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -1 \\ -4 & 3 \end{bmatrix}$$

b. i. Calculate $\det(B)$.

$$\det(B) = 2 \times 7 - 5 \times 3 = 14 - 15 = -1$$

ii. Find $B^{-1}$ (if it exists).

$$B^{-1} = -1 \begin{bmatrix} 7 & -5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix}$$

c. i. Calculate $\det(C)$.

$$\det(C) = 8 \times 2 - 2 \times 8 = 16 - 16 = 0$$

ii. Find $C^{-1}$ (if it exists).

Since $\det(C) = 0$, $C^{-1}$ doesn't exist.
If the determinant of \(
\begin{vmatrix}
5 & -3 \\
x & 3
\end{vmatrix}
\) is equal to 21, what is the value of \(x\)?

For the matrix \(A = \begin{bmatrix} 2 & 3 \\ x & -4 \end{bmatrix}\), what is the value of \(x\) for which \(A = A^{-1}\)?

**Working**

\(\text{a} \quad 1\) Find the determinant of the matrix. 
\[
\det = 5 \times 3 - (-3) \times x = 15 + 3x
\]

\(\text{b} \quad 1\) Find the determinant of \(A\). 
\[
\det(A) = 2 \times (-4) - 3 \times x = -8 - 3x
\]

\(\text{a} \quad 2\) Let the determinant equal the number given and solve for \(x\), using a CAS/calculator if necessary.
\[
15 + 3x = 21 \\
3x = 21 - 15 = 6 \\
x = 2
\]

\(\text{b} \quad 2\) Find \(A^{-1}\).
\[
A^{-1} = \frac{1}{-8 - 3x} \begin{bmatrix} -4 & -3 \\ -x & 2 \end{bmatrix}
\]
\[
= \begin{bmatrix} -4 & -3 \\ -8 - 3x & -8 - 3x \\ -x & 2 \\ -8 - 3x & -8 - 3x \end{bmatrix}
\]

\(\text{a} \quad 3\) Let \(A = A^{-1}\) and, using the fact that for matrices to be equal every corresponding element has to be equal, set 2 corresponding elements equal to each other.

\[
A = A^{-1} \begin{bmatrix} 2 & 3 \\ x & -4 \end{bmatrix}
\]
\[
= \begin{bmatrix} -4 & -3 \\ -8 - 3x & -8 - 3x \\ -x & 2 \\ -8 - 3x & -8 - 3x \end{bmatrix}
\]

\[
\text{So,} \\
2 = \frac{-4}{-8 - 3x}
\]

\[
2(-8 - 3x) = -4 \\
-16 - 6x = -4 \\
6x = -12 \\
x = -2
\]
Using CAS Finding the determinant and inverse of a matrix

For \( A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \), find \( \det(A) \) and \( A^{-1} \).

**STEP 1**
On a calculator page press \( \text{ENTRY} \).

**STEP 2**
Create a 3 × 3 matrix and enter the elements for matrix \( A \) above. Press \( \text{ENTRY} \) to store the matrix as \( a \).

**STEP 3**
Press \( \text{menu} \);
7 Matrix & Vector
3 Determinant

Press \( \text{ENTRY} \).

**STEP 4**
To calculate the inverse \( a^{-1} \);
Type \( \text{ENTRY} \)
To approximate enter \( a^{-1} \) and press \( \text{ENTRY} \)
STEP 5
Write the answer.

\[
\text{det} (A) = -5
\]

\[
A^{-1} = \frac{1}{5} \begin{bmatrix}
1 & -1 & 1 \\
-2 & -3 & 3 \\
3 & 7 & -2
\end{bmatrix} = \begin{bmatrix}
0.2 & -0.2 & 0.2 \\
-0.4 & -0.6 & 0.6 \\
0.6 & 1.4 & -0.4
\end{bmatrix}
\]

CLASSPAD

STEP 1
Use the \( \text{Main} \) application. Press \( \text{Keyboard} \), then \( \text{Math2} \) and tap on \( \text{M} \) twice for a \( 3 \times 3 \) matrix.

STEP 2
Enter the numbers in the matrix by tapping each square and typing the number. You can also use the arrows to move around the elements of the matrix.

Assign a name to the matrix by tapping \( \text{M} \), then the name. Press \( \text{EX} \).

STEP 3
To find the determinant, highlight \( A \) that has been defined, then tap \( \text{Interactive} \). Matrix, Calculation, \( \text{det} \) to get \( \text{det} (A) = -5 \)
STEP 4
To calculate the inverse of $A$: $A^{-1}$, type $A^\wedge (1)$.

Press \text{[inv]}.

STEP 5
Write the answer.

\[
\det(A) = -5
\]

\[
A^{-1} = \begin{bmatrix}
0.2 & -0.2 & 0.2 \\
-0.4 & -0.6 & 0.6 \\
0.6 & 1.4 & -0.4 \\
\end{bmatrix}
\]

EXAM PREP 7.4

**Inverse matrices**

**Prep 1**  WORKED EXAMPLE 8

If $A = \begin{bmatrix} 5 & 2 \\ 1 & 11 \end{bmatrix}$ and $I$ is the identity matrix, find $AI$ and $IA$ by hand.

What do you notice when you compare the two answers?

**Prep 2**  WORKED EXAMPLE 9

Show that the matrices $\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ are inverses of each other.

**Prep 3**  WORKED EXAMPLE 10

For each of the following matrices, find

- \text{i} the determinant
- \text{ii} the inverse (if it exists).

\begin{align*}
ap & \quad A = \begin{bmatrix} 2 & 8 \\ 2 & 7 \end{bmatrix} & \quad bp & \quad B = \begin{bmatrix} 5 & 10 \\ 1 & 1 \end{bmatrix} & \quad cp & \quad C = \begin{bmatrix} 5 & 2 \\ 10 & 4 \end{bmatrix}
\end{align*}

**Prep 4**  WORKED EXAMPLE 11

\begin{align*}
ap & \quad \text{If the determinant of } \begin{bmatrix} 6 & -12 \\ 3 & x \end{bmatrix} \text{ is equal to 12, what is the value of } x? \\
bp & \quad \text{For the matrix } A = \begin{bmatrix} 5 & x \\ -4 & -5 \end{bmatrix}, \text{ what is the value of } x \text{ for which } A = A^{-1}?
\end{align*}
Prep 5

USING CAS: FINDING THE DETERMINANT AND INVERSE OF A MATRIX

For each of the following matrices, find

i \( \det (A) \)  
ii \( A^{-1} \)

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>A =</td>
<td>[ \begin{bmatrix} 4 &amp; 3 &amp; 6 \ 2 &amp; -10 &amp; 7 \ -1 &amp; -8 &amp; 1 \end{bmatrix} ]</td>
<td>b</td>
<td>A =</td>
</tr>
<tr>
<td>c</td>
<td>A =</td>
<td>[ \begin{bmatrix} 5 &amp; 1 &amp; 4 \ 0 &amp; -2 &amp; 1 \ -1 &amp; -4 &amp; 1 \end{bmatrix} ]</td>
<td>d</td>
<td>A =</td>
</tr>
</tbody>
</table>

EXAM PRACTICE 7.4

Inverse matrices

**Question 1**

What is the inverse of the matrix \[ \begin{bmatrix} 2 & -1 \\ 4 & 8 \end{bmatrix} \]?

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>[ \begin{bmatrix} 1 &amp; -1 \ 10 &amp; 20 \ 1 &amp; 2 \ 5 &amp; 5 \end{bmatrix} ]</td>
<td>B</td>
<td>[ \begin{bmatrix} 2 &amp; -1 \ 1 &amp; 4 \ 4 &amp; 8 \end{bmatrix} ]</td>
<td>C</td>
</tr>
<tr>
<td>D</td>
<td>[ \begin{bmatrix} 1 &amp; -2 \ 20 &amp; 4 \ -1 &amp; 5 \ -8 \end{bmatrix} ]</td>
<td>E</td>
<td>[ \begin{bmatrix} 8 &amp; 1 \ 12 &amp; -4 \ 2 \end{bmatrix} ]</td>
<td></td>
</tr>
</tbody>
</table>

**Question 2**

What is \( \det (M) \) if \( M = \begin{bmatrix} 3 & 6 \\ -1 & 5 \end{bmatrix} \)?

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>13</td>
<td>B</td>
<td>[ \frac{1}{21} ]</td>
<td>C</td>
</tr>
</tbody>
</table>

**Question 3**

Which of the following is not true for the \( 2 \times 2 \) identity matrix, \( I \)?

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( \det (I) = 1 )</td>
<td>B</td>
<td>( I + I = I )</td>
<td>C</td>
</tr>
</tbody>
</table>

**Question 4**

Which of the following matrices has no inverse?

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>[ \begin{bmatrix} 1 &amp; 3 \ 3 &amp; 1 \end{bmatrix} ]</td>
<td>B</td>
<td>[ \begin{bmatrix} 10 &amp; 4 \ 5 &amp; 2 \end{bmatrix} ]</td>
<td>C</td>
</tr>
</tbody>
</table>
Question 5
The inverse for the matrix \[
\begin{bmatrix}
1 & x \\
-1 & 2
\end{bmatrix}
\] is

A \[
\begin{bmatrix}
2 & -x \\
1 & 1
\end{bmatrix}
\]

B \[
\frac{1}{2+x} \begin{bmatrix}
2 & -x \\
1 & 1
\end{bmatrix}
\]

C \[
\frac{1}{2-x} \begin{bmatrix}
2 & -x \\
1 & 1
\end{bmatrix}
\]

D \[
\frac{1}{2+x} \begin{bmatrix}
2 & x \\
-1 & 1
\end{bmatrix}
\]

E \[
\frac{1}{2-x} \begin{bmatrix}
2 & x \\
1 & 1
\end{bmatrix}
\]

Question 6
If \(A\) is a 2 \(\times\) 3 matrix, \(B\) is a 3 \(\times\) 2 matrix and \(C\) is a 3 \(\times\) 3 matrix, which one of the following matrices can you be certain has no inverse?

A \(BA\)  B \(C\)  C \(C^2\)  D \(AB\)  E \(CB\)

Question 7
For a 3 \(\times\) 3 matrix, \(A\), with a determinant of \(-7\), which of the following statements is false?

A \(A^2\) is a 3 \(\times\) 3 matrix  B \(AA^{-1} = I\)  C \(\det(A) = \det(IA)\)

D \(A\) is a square matrix.  E \(A^{-1}\) is not defined.

Question 8
The determinant of \[
\begin{bmatrix}
3 & 2 \\
6 & x
\end{bmatrix}
\] is equal to 9. What is the value of \(x\)?

A \(-7\)  B \(-4.5\)  C \(1\)  D \(4.5\)  E \(7\)

[VCAA 2008 1MQ5]

Question 9
Matrix \(A\) is a 1 \(\times\) 3 matrix. Matrix \(B\) is a 3 \(\times\) 1 matrix.
Which one of the following matrix expressions involving \(A\) and \(B\) is defined?

A \(A + \frac{1}{3}B\)  B \(2B \times 3A\)  C \(A^2B\)  D \(B^{-1}\)  E \(B - A\)

[VCAA 2008 1MQ4]

Question 10
Matrix \(A\) is a 3 \(\times\) 4 matrix. Matrix \(B\) is a 3 \(\times\) 3 matrix.
Which one of the following matrix expressions is defined?

A \(BA^2\)  B \(BA - 2A\)  C \(A + 2B\)  D \(B^2 - AB\)  E \(A^{-1}\)

[VCAA 2011 1MQ4]
Matrices are often a convenient way of representing real-life information and matrix arithmetic can be used to do relevant calculations. The advantage of matrices when you use a CAS/calculator is that they allow you to do a large number of calculations quickly.

Each week, the coach of the ‘Little Diggers’ basketball team awards the ‘Best and Fairest Player Award’ to the player who scores the most game points. The results of their last game were:

<table>
<thead>
<tr>
<th></th>
<th>3 pointers</th>
<th>2 pointers</th>
<th>1 pointer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asma</td>
<td>0</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>Sarah</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Lucy</td>
<td>0</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>Matt</td>
<td>2</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>Kelly</td>
<td>1</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>Min-lee</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Sophie</td>
<td>1</td>
<td>8</td>
<td>0</td>
</tr>
</tbody>
</table>

a Write a column matrix, $P$, to represent the 3 different scores possible.

b Write a matrix, $G$, to represent the data from the table.

c Use matrix multiplication to find a score matrix named $S$ that represents the total scored by each of the players.

d Which player won the ‘Best and Fairest’ award?

e The opposition team scored a total of 87 points. Did the Little Diggers win the game?

**Working**

a When a player scores, they can either get 3 points, 2 points or 1 point.

$$P = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

b Matrix $G$ will have 7 rows and 3 columns.

$$G = \begin{bmatrix} 0 & 5 & 0 \\ 1 & 3 & 2 \\ 0 & 7 & 0 \\ 2 & 5 & 0 \\ 1 & 7 & 0 \\ 0 & 2 & 2 \\ 1 & 8 & 0 \end{bmatrix}$$
To find \( S \), we must find \( GP \).

This will give us \((7 \times 3) \times (3 \times 1)\), which will result in a \(7 \times 1\) matrix that represents the personal total for each of the seven players.

\[
S = \begin{bmatrix}
0 & 5 & 0 \\
1 & 3 & 2 \\
0 & 7 & 0 \\
2 & 5 & 0 \\
1 & 7 & 0 \\
0 & 2 & 2 \\
1 & 8 & 0 \\
\end{bmatrix}
\begin{bmatrix}
3 \\
1 \\
2 \\
1 \\
1 \\
2 \\
1 \\
\end{bmatrix}
= \begin{bmatrix}
10 \\
11 \\
14 \\
16 \\
17 \\
6 \\
19 \\
\end{bmatrix}
\]

The largest element in \( S \) is the highest personal score.

Sophie won with 19 points.

The total score for Little Diggers can be found by adding all of the elements in \( S \).

Score = 10 + 11 + 14 + 16 + 17 + 6 + 19 = 93

Write the answer.

93 is greater than 87, so Little Diggers won.
The manager of a local hardware store purchases small metal fasteners for $3 each and large metal fasteners for $5. In the last two weeks, he has purchased the number of fasteners as shown.

a. Find the two matrices that can be multiplied to give the total purchase cost of metal fasteners in each of the two weeks.

The manager sells goods at a 55% mark-up. He recorded his purchase costs over the last 2 weeks for metal fasteners and three other items in this table.

b. i. Represent these costs in a $4 \times 2$ cost matrix, $C$.

ii. The selling price is calculated by finding 155% of the cost price. Convert 155% to a decimal.

iii. Using scalar multiplication, represent the selling prices of these goods in a $4 \times 2$ matrix, $S$.

c. i. Create a profit matrix.

ii. Calculate the total profit to be made if all of the goods purchased over these 2 weeks are sold.
To create a profit matrix:

\[
\text{Profit} = \text{Selling price} - \text{cost price}
\]

\[
\begin{bmatrix}
1643.00 & 2410.25 \\
4694.95 & 1742.20 \\
1430.65 & 1092.75 \\
1388.00 & 1906.50 \\
\end{bmatrix} - 
\begin{bmatrix}
1060 & 1555 \\
3029 & 1124 \\
923 & 705 \\
896 & 1230 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
583.00 & 855.25 \\
1665.95 & 618.20 \\
507.65 & 387.75 \\
492.80 & 676.50 \\
\end{bmatrix}
\]

Total profit can be found by adding all of the elements in the profit matrix.

Total profit =  583.00 + 855.25 + 1665.95 + 618.20 + 507.65 + 387.75 + 492.80 + 676.50

= $5787.10

Matrix B shows the number of sightings of rare birds made by members of a bird-watching club.

\[
B = \begin{bmatrix}
13 & 3 & 4 \\
2 & 5 & 2 \\
1 & 1 & 3 \\
5 & 7 & 10 \\
\end{bmatrix}
\]

\[
P = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}
\]

\[
Q = \begin{bmatrix}
1 \\
1 \\
\end{bmatrix}
\]

Calculate the product \( A = PB \). What information is given by this matrix?

What information does the element \( a_{13} \) give?

Calculate the product \( C = BQ \). What information is given by this matrix?

What information does the element \( c_{21} \) give?

**Working**

Do the multiplication. The matrix order equation tells us the product will be \((1 \times 4) \times (4 \times 3) = (1 \times 3)\).

\[
A = PB \\
= \begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 5 & 2 & 1 \\
1 & 1 & 3 & 1 \\
5 & 7 & 10 & 1 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
13 & 3 & 4 \\
2 & 5 & 2 \\
1 & 1 & 3 \\
5 & 7 & 10 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
21 & 16 & 19 \\
\end{bmatrix}
\]

The product involves summing each type of bird. \( A \) gives the totals for each type of bird sighted by the club on the weekend.

\( PB \) is the element in the \( i \)th row and \( j \)th column in the product \( PB \). \( a_{13} \) tells us that there were a total of 19 redtwits sighted on the weekend.
b  

i  
Do the multiplication. The matrix order equation tells us the product will be 
\((4 \times 3) \times (3 \times 1) = (4 \times 1)\).

\[
C = BQ = \begin{bmatrix}
13 & 3 & 4 \\
2 & 5 & 2 \\
1 & 1 & 3 \\
5 & 7 & 10
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
= \begin{bmatrix}
20 \\
9 \\
5 \\
22
\end{bmatrix}
\]

The product involves summing the birds that each bird watcher sighted.

ii  
\((BQ)_{ij}\) is the element in the \(i\)th row and \(j\)th column in the product \(BQ\).

\(c_{21}\) tells us that Esther sighted a total of 9 birds on the weekend.

---

**Exam hack**

Even if your answer for a matrix multiplication is a single number, you must give your answer with matrix brackets around it, or it won’t be marked as correct.

---

**Using matrices**

**Prep 1**  
**WORKED EXAMPLE 12**

Each week the coach of the ‘Little Clunkers’ cricket team awards the ‘Best Batter’ to the player who makes the most runs. The results of their last game are shown in the table.

- **a** Write a column matrix, \(S\), to represent the 3 different ways of scoring runs.
- **b** Write a matrix, \(R\), to represent the data from the table.
- **c** Use matrix multiplication to find a score matrix, named \(S\), which represents the total runs made by each of the players.
- **d** Which player won the ‘Best Batter’ award?
- **e** The opposition team made 207 runs. Did the Little Clunkers win the game?

<table>
<thead>
<tr>
<th>Player</th>
<th>Single runs</th>
<th>Fours</th>
<th>Sixes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heshan</td>
<td>26</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Sam</td>
<td>12</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Ahmat</td>
<td>18</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>Toni</td>
<td>9</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Vishna</td>
<td>30</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>Jordan</td>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Yasara</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

---

---
A chemist purchases small bottles of vitamin C tablets for $2 each and large bottles of vitamin C tablets for $3. In the last two weeks, he has purchased the number of vitamin C bottles shown in the table.

<table>
<thead>
<tr>
<th>Week</th>
<th>Small bottles of vitamin C</th>
<th>Large bottles of vitamin C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week 1</td>
<td>75</td>
<td>60</td>
</tr>
<tr>
<td>Week 2</td>
<td>47</td>
<td>82</td>
</tr>
</tbody>
</table>

a Find the two matrices that can be multiplied to give the total purchase cost of vitamin C bottles in each of the two weeks.

The chemist sells goods at a 75% mark-up. He recorded his purchase costs over the last 2 weeks for vitamin C tablets and three other items in the table.

b i Represent these costs in a $4 \times 2$ cost matrix, $C$.

ii The selling price is calculated by finding 175% of the cost price. Convert 175% to a decimal.

iii Using scalar multiplication, represent the selling prices of these goods in a $4 \times 2$ matrix, $S$.

c i Create a profit matrix.

ii Calculate the total profit to be made if all of the goods purchased over these 2 weeks are sold.

Matrix $B$ shows the number of different trains seen by members of a trainspotting club over a weekend.

a i Calculate the product $A = PB$.

ii What information is given by this matrix?

iii What information does the element $a_{12}$ give?

b i Calculate the product $C = BQ$.

ii What information is given by this matrix?

iii What information does the element $c_{21}$ give?
Using matrices

Question 1

Five students, Richard (R), Brendon (B), Lee (L), Arif (A) and Karl (K), were asked whether they played each of the following sports, football (F), golf (G), soccer (S) or tennis (T). Their responses are displayed in the table.

<table>
<thead>
<tr>
<th>Student</th>
<th>Football (F)</th>
<th>Golf (G)</th>
<th>Soccer (S)</th>
<th>Tennis (T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>B</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>L</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>A</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>K</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>yes</td>
</tr>
</tbody>
</table>

If 1 is used to indicate that the student plays a particular sport and 0 is used to indicate that the student does not play a particular sport, which matrix represents the information in the table?
Question 2

A company makes Regular (R), Queen (Q) and King (K) size beds. Each bed comes in either the Classic style or the more expensive Deluxe style. The price of each style of bed, in dollars, is listed in a price matrix $P$, shown.

The company wants to increase the price of all beds.

A new price matrix, listing the increased prices of the beds, can be generated from $P$ by forming a matrix product with the matrix, $M$, shown.

This new price matrix is

- A
- B
- C
- D
- E

[VCAA 2006 1MQ5]

Question 3

The cost prices of three different electrical items in a store are $230, $290 and $310 respectively.

The selling price of each of these three electrical items is 1.3 times the cost price plus a commission of $20 for the salesman.

A matrix that lists the selling price of each of these three electrical items is determined by evaluating

- A
- B
- C
- D
- E

[VCAA 2008 1MQ3]

Question 4

Peter bought only apples and bananas from his local fruit shop. The matrix $N = \begin{bmatrix} 3 & 4 \end{bmatrix}$ lists the number of apples ($A$) and bananas ($B$) that Peter bought. The matrix $C = \begin{bmatrix} 0.37 & A \\ 0.43 & B \end{bmatrix}$ lists the cost (in dollars) of one apple and one banana respectively. The matrix product, $NC$, gives

- A
- B
- C
- D
- E

[VCAA 2010 1MQ2]
Question 5

Apples cost $3.50 per kg, bananas cost $4.20 per kg and carrots cost $1.89 per kg. Ashley buys 3 kg of apples, 2 kg of bananas and 1 kg of carrots. A matrix product to calculate the total cost of these items is

\[
A = \begin{bmatrix}
3 & 3.50 \\
2 & 4.20 \\
1 & 1.89
\end{bmatrix} \quad B = \begin{bmatrix}
3 & 2 & 1 \\
3.50 & 4.20 & 1.89
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
3.50 \times 2 & 4.20 \times 3 & 1.89 \times 1
\end{bmatrix} \quad D = \begin{bmatrix}
3 & 2 & 1 \\
3.50 & 4.20 & 1.89
\end{bmatrix}
\]

\[
E = \begin{bmatrix}
3.50 & 4.20 & 1.89 \\
3 & 2 & 1
\end{bmatrix}
\]

[VCAA 2008 1MQ2]

Question 6

Three types of cheese, Cheddar (C), Gouda (G) and Blue (B), will be bought for a school function. The cost matrix \( P \) lists the prices of these cheeses, in dollars, at two stores, Foodway and Safeworth.

a What is the order of matrix \( P \)?

b i Evaluate the matrix \( W = PQ \).

ii At which store will the total cost of the cheese be lower?

[VCAA 2009 2MQ1]

Question 7

Two subjects, Biology and Chemistry, are offered in the first year of a university science course. The matrix \( N \) lists the number of students enrolled in each subject. The matrix \( P \) lists the proportion of these students expected to be awarded an A, B, C, D or E grade in each subject.

\[
N = \begin{bmatrix}
460 & 360
\end{bmatrix} \quad \begin{array}{c}
\text{Biology} \\
\text{Chemistry}
\end{array} \quad P = \begin{bmatrix}
0.05 & 0.125 & 0.175 & 0.45 & 0.20
\end{bmatrix} 
\]

a Write down the order of matrix \( P \).

b Let the matrix \( R = NP \).

i Evaluate the matrix \( R \).

ii Explain what the matrix element \( R_{24} \) represents.

[c Students enrolled in Biology have to pay a laboratory fee of $110, while students enrolled in Chemistry pay a laboratory fee of $95.

i Write down a clearly labelled row matrix, called \( F \), that lists these fees.

ii Show a matrix calculation that will give the total laboratory fees, \( L \), paid in dollars by the students enrolled in Biology and Chemistry. Find this amount.

[VCAA 2008 2MQ1]
Question 8

A manufacturer sells three products, A, B and C, through outlets at two shopping centres, Eastown (E) and Noxland (N). The number of units of each product sold per month through each shop is given by the matrix $Q$.

\[ Q = \begin{bmatrix}
2500 & 3400 & 1890 \\
1765 & 4588 & 2456 \\
\end{bmatrix} \]

a  Write down the order of matrix $Q$.  

The matrix $P$ shown gives the selling price, in dollars, of products A, B, C.

\[ P = \begin{bmatrix}
14.50 & A \\
21.60 & B \\
19.20 & C \\
\end{bmatrix} \]

b  i  Evaluate the matrix $M$, where $M = QP$.  

ii  What information do the elements of matrix $M$ provide?  

c  Explain why the matrix $PQ$ is not defined. 

[VCAA 20062MQ1]

Question 9

Rosa uses the following six-digit pin number for her bank account: 216342. With her knowledge of matrices, she decides to use matrix multiplication to disguise this pin number. First, she writes the six digits in the $2 \times 3$ matrix $A$.

\[ A = \begin{bmatrix}
2 & 6 & 4 \\
1 & 3 & 2 \\
\end{bmatrix} \]

Next, she creates a new matrix by forming the matrix product, $C = BA$, where $B = \begin{bmatrix}
1 & -1 \\
2 & -1 \\
\end{bmatrix}$.

a  i  Determine the matrix $C = BA$.  

ii  From the matrix $C$, Rosa is able to write down a six-digit number that disguises her original pin number. She uses the same pattern that she used to create matrix $A$ from the digits 216342.

Write down the new six-digit number that Rosa uses to disguise her pin number.  

b  Show how the original matrix $A$ can be regenerated from matrix $C$.  

[VCAA 2012 2MQ2]
Question 10

To reduce the number of insects in a wetland, the wetland is sprayed with an insecticide. The numbers of insects \( I \), birds \( B \), lizards \( L \) and frogs \( F \) in the wetland that have been sprayed with insecticide are displayed in the matrix \( N \).

Unfortunately, the insecticide that is used to kill the insects can also kill birds, lizards and frogs. The proportions of insects, birds, lizards and frogs that have been killed by the insecticide are displayed in the matrix \( D \).

\[
D = \begin{bmatrix}
0.995 & 0 & 0 & 0 \\
0 & 0.05 & 0 & 0 \\
0 & 0 & 0.025 & 0 \\
0 & 0 & 0 & 0.30
\end{bmatrix}
\]

Alive before spraying

\[
I \quad B \quad L \quad F
\]

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0.995</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>0.05</td>
<td>0</td>
</tr>
<tr>
<td>L</td>
<td>0</td>
<td>0</td>
<td>0.025</td>
</tr>
<tr>
<td>F</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Dead after spraying

\[
I \quad B \quad L \quad F
\]

a Evaluate the matrix product \( K = ND \). 1 mark

b Use the information in matrix \( K \) to determine the number of birds that have been killed by the insecticide. 1 mark

c Evaluate the matrix product \( M = KF \), where \( F = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \). 1 mark

d In the context of the problem, what information does matrix \( M \) contain? 1 mark

Question 11

In a game of basketball, a successful shot for goal scores one point, two points, or three points, depending on the position from which the shot is thrown. \( G \) is a column matrix that lists the number of points scored for each type of successful shot.

In one game, Oscar was successful with 4 one-point shots for goal, 8 two-point shots for goal, and 2 three-point shots for goal.

a Write a row matrix, \( N \), that shows the number of each type of successful shot for goal that Oscar had in that game. 1 mark

b Matrix \( P \) is found by multiplying matrix \( N \) with matrix \( G \) so that \( P = N \times G \). Evaluate matrix \( P \). 1 mark

c In this context, what does the information in matrix \( P \) provide? 1 mark
Solving two simultaneous equations using matrices

Matrices can be used to solve simultaneous equations.

**Worked example 15**

**a** Show that the matrix equation
\[
\begin{bmatrix}
5 & 3 \\
6 & 4
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
13 \\
16
\end{bmatrix}
\]
generates the simultaneous equations
\[
5x + 3y = 13 \\
6x + 4y = 16
\]

**b** Solve the simultaneous equations using the matrix equation, showing all the steps.

**Working**

1. Multiply the matrices on the left of the equation.
\[
\begin{bmatrix}
5 & 3 \\
6 & 4
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
13 \\
16
\end{bmatrix}
\]

2. Equate the elements.
\[
5x + 3y = 13 \\
6x + 4y = 16
\]

1. Find the inverse of the square matrix in the matrix equation.

   The inverse of
   \[
   \begin{bmatrix}
   5 & 3 \\
   6 & 4
   \end{bmatrix}
   \]

   is
   \[
   \frac{1}{20-18}
   \begin{bmatrix}
   4 & -3 \\
   -6 & 5
   \end{bmatrix}
   = \frac{1}{2}
   \begin{bmatrix}
   4 & -3 \\
   -6 & 5
   \end{bmatrix}
   =
   \begin{bmatrix}
   2 & -1.5 \\
   -3 & 2.5
   \end{bmatrix}
   \]

2. Pre-multiply both sides of the matrix equation by the inverse.
\[
\begin{bmatrix}
2 & -1.5 \\
-3 & 2.5
\end{bmatrix}
\begin{bmatrix}
5 & 3 \\
6 & 4
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
2 & -1.5 \\
-3 & 2.5
\end{bmatrix}
\begin{bmatrix}
13 \\
16
\end{bmatrix}
\]

3. Use the fact that a matrix multiplied by its inverse is the identity matrix.
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
2 & -1.5 \\
-3 & 2.5
\end{bmatrix}
\begin{bmatrix}
13 \\
16
\end{bmatrix}
\]

4. A matrix multiplied by the identity matrix leaves the matrix unchanged.
\[
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
2 & -1.5 \\
-3 & 2.5
\end{bmatrix}
\begin{bmatrix}
13 \\
16
\end{bmatrix}
\]

5. Find the product on the right of the equation.
\[
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
26 - 24 \\
-39 + 40
\end{bmatrix} =
\begin{bmatrix}
2 \\
1
\end{bmatrix}
\]

6. Equate the elements to solve the simultaneous equations.
\[
x = 2, y = 3
\]
Inconsistent and dependent systems of equations without solutions

Not every set of simultaneous equations has a unique solution.

The lines represented by the equations may be parallel. For example:

\[ 2x - y = 5 \text{ (or } y = 2x - 5) \]
\[ 2x - y = -3 \text{ (or } y = 2x + 3) \]

We can see they both have a slope of 2 and hence no intersection. This means the simultaneous equations have no solution. They are called an inconsistent system of equations.

The lines represented by the equations may actually be the same line. For example:

\[ 2x - y = 5 \]
\[ 4x - 2y = 10 \]

If we divide both sides of the second equation by 2, then we get \(2x - y = 5\), which is the same as the first equation. This means there are an infinite number of points of intersection between the two lines, so there is no unique solution. They are called a dependent system of equations.

When solving simultaneous equations using matrices, if there is no unique solution, you will get a determinant of zero for the square matrix, which means no inverse exists. If you are using a CAS/calculator it will give you an error message: either ‘Singular matrix’ or ‘Undefined’.
For each pair of simultaneous equations, use matrices to find the value of $e$ for which the equations do not have a unique solution.

**a**  
\[\begin{align*} 
7x + 5y &= 12 \\
ex + 5y &= 11 
\end{align*}\]

**b**  
\[\begin{align*} 
11x - 3y &= -9 \\
22x + ey &= 2 
\end{align*}\]

**Working**

**a**  
1. Write the simultaneous equations in matrix form.
\[
\begin{bmatrix} 7 & 5 \\ e & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 11 \end{bmatrix}
\]

2. Find the determinant.
\[
\text{det} = 7 \times 5 - 5 \times e = 35 - 5e
\]

3. Let the determinant equal zero and solve, using a CAS/calculator if necessary.
\[
35 - 5e = 0 \\
5e = 35 \\
e = 7
\]

**b**  
1. Write the simultaneous equations in matrix form.
\[
\begin{bmatrix} 11 & -3 \\ 22 & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -9 \\ 2 \end{bmatrix}
\]

2. Find the determinant.
\[
\text{det} = 11 \times e - (-3) \times 22 = 11e + 66
\]

3. Let the determinant equal zero and solve for $e$, using a CAS/calculator if necessary.
\[
11e + 66 = 0 \\
11e = -66 \\
e = -6
\]
Solving three or more simultaneous equations using matrices

The matrix method is extremely effective when solving three or more simultaneous equations.

Using CAS

Solving three or more simultaneous equations using matrices

Solve the following using matrices.

\[ 2x + 3y - z = 7 \]
\[ 3x + 2z = 2 \]
\[ x + y + z = 7 \]

**STEP 1**

Write the simultaneous equations as a matrix equation. Insert zeros where a pronumeral is missing.

\[
\begin{bmatrix}
2 & 3 & -1 \\
3 & 0 & 2 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
7 \\
2 \\
7
\end{bmatrix}
\]

**STEP 2**

Identify the matrices and write them in the form \( AX = B \)

\[
A = \begin{bmatrix}
2 & 3 & -1 \\
3 & 0 & 2 \\
1 & 1 & 1
\end{bmatrix}, 
X = \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}, 
B = \begin{bmatrix}
7 \\
2 \\
7
\end{bmatrix}
\]

\[ AX = B \]

**STEP 3**

As long as \( \det(A) \neq 0 \), the solution is found by pre-multiplying both sides by \( A^{-1} \).

\[ X = A^{-1}B \]

**STEP 4**

Enter \( A \) and \( B \) into your CAS/calculator.

**TI-NSPIRE CAS**

Create a \( 3 \times 3 \) matrix, enter the elements shown and store as \( a \).

Create a \( 3 \times 1 \) matrix, enter the elements shown and store as \( b \).
**STEP 5**

Find the product $a^{-1}b$.

**STEP 6**

Write your answer.

$$X = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}$$

$x = -2$, $y = 5$ and $z = 4$. 
A factory has three robots which assemble three different models of computers (A, B and C).

Robot 1 assembles three model A's, two model B's and five model C's in 113 minutes.
Robot 2 assembles one model A and three model B's in 56 minutes.
Robot 3 assembles two model B's and one model C in 40 minutes.

Let $a =$ the amount of time in minutes it takes for a robot to assemble one of model A.
Let $b =$ the amount of time in minutes it takes a robot to assemble one of model B.
Let $c =$ the amount of time in minutes it takes a robot to assemble one of model C.

**a** Write three simultaneous equations in terms of $a$, $b$ and $c$.

**b** Write the simultaneous equations in matrix form.

**c** Solve the matrix equation and hence find how long it takes a robot to assemble each of the three computer models.

---

**Working**

**a** Use the information in the question to write three simultaneous equations.

\[
\begin{align*}
3a + 2b + 5c &= 113 \\
3a + 3b &= 56 \\
2b + c &= 40
\end{align*}
\]

**b** Rewrite in matrix form, adding zeros where necessary.

\[
\begin{bmatrix}
3 & 2 & 5 \\
1 & 3 & 0 \\
0 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
=
\begin{bmatrix}
113 \\
56 \\
40
\end{bmatrix}
\]

**c** Solve with a CAS/calculator by finding the inverse of the $3 \times 3$ matrix and pre-multiplying the inverse on both sides of the equation.

\[
\begin{align*}
\frac{1}{17}
\begin{bmatrix}
3 & 8 & -15 \\
-1 & 3 & 5 \\
2 & -6 & 7
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
&=
\begin{bmatrix}
113 \\
56 \\
40
\end{bmatrix}
\end{align*}
\]

**2** Write the answer.

It takes a robot 11 minutes to assemble model A, 15 minutes to assemble model B, and 10 minutes to assemble model C.
Matrices and simultaneous equations

**Prep 1 WORKED EXAMPLE 15**

a Show that the matrix equation \[
\begin{bmatrix}
4 & -1 \\
1 & -3
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
-5 \\
7
\end{bmatrix}
\] generates the simultaneous equations
\[4x - y = -5, \quad x - 3y = 7\]

b Solve the simultaneous equations using the matrix equation, showing all the steps.

**Prep 2 WORKED EXAMPLE 16**

For each pair of simultaneous equations, use matrices to find the value of \(e\) for which the equations do not have a unique solution.

a \[6x - 4y = 16, \quad ex - 4y = 13\]

b \[12x + 3y = -3, \quad 4x + ey = -1\]

**Prep 3 USING CAS: SOLVING THREE OR MORE SIMULTANEOUS EQUATIONS USING MATRICES**

Solve each of the following using matrices.

a \[5x - y + 2z = 1, \quad -x - 3y = -5, \quad x + 5z = 19\]

b \[x + y + z = 12, \quad 2x + 2y - 3z = 9\]

b \[10x - 5y = -40, \quad 2x + 3z = 7\]

**Prep 4 WORKED EXAMPLE 17**

A worker assembles a particular model of laptop, printer, modem and router.

She assembles

- two laptops, five printers, three modems and one router in 167 minutes
- four laptops and ten printers in 240 minutes
- six printers, twelve modems and three routers in 273 minutes
- five modems and two routers in 82 minutes

Let \(a\) = the amount of time in minutes it takes her to assemble one laptop.

Let \(b\) = the amount of time in minutes it takes her to assemble one printer.

Let \(c\) = the amount of time in minutes it takes her to assemble one modem.

Let \(d\) = the amount of time in minutes it takes her to assemble one router.

a Write four simultaneous equations in terms of \(a\), \(b\), \(c\) and \(d\).

b Write the simultaneous equations in matrix form.

c Solve the matrix equation and hence find how long it takes her to assemble each of the four items.
Matrices and simultaneous equations

Question 1

The matrix equation
\[
\begin{bmatrix}
4 & 2 & 8 \\
2 & 0 & 3 \\
0 & 3 & -1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
7 \\
2 \\
6
\end{bmatrix}
\]
can be used to solve the system of simultaneous linear equations

\[
\begin{align*}
4x + 2y + 8z &= 7 \\
2x + 3y &= 2 \\
3x - y &= 6
\end{align*}
\]

\[
\begin{align*}
4x + 2y + 8z &= 7 \\
2x + 3y &= 2 \\
3x - z &= 6
\end{align*}
\]

\[
\begin{align*}
4x + 2y + 8z &= 7 \\
2x + 3y &= 2 \\
3x - z &= 6
\end{align*}
\]

\[
\begin{align*}
4x + 2y + 8z &= 7 \\
2x + 3y &= 2 \\
3x - z &= 6
\end{align*}
\]

\[
\begin{align*}
4x + 2y + 8z &= 7 \\
2x + 3y &= 2 \\
3x - z &= 6
\end{align*}
\]

Question 2

The solution of the matrix equation
\[
\begin{bmatrix}
0 & -3 & 2 \\
1 & 1 & 1 \\
-2 & 0 & 3
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
11 \\
5 \\
8
\end{bmatrix}
\]
is

\[
\begin{bmatrix}
1 \\
24 \\
2
\end{bmatrix}
\begin{bmatrix}
2 \\
-1 \\
4
\end{bmatrix}
\begin{bmatrix}
2 \\
1 \\
3
\end{bmatrix}
\begin{bmatrix}
-11 \\
4 \\
3
\end{bmatrix}
\begin{bmatrix}
11 \\
5 \\
8
\end{bmatrix}
\]

Question 3

A system of three simultaneous linear equations is written in matrix form as follows.

\[
\begin{bmatrix}
1 & -2 & 0 \\
1 & 0 & 3 \\
0 & 2 & -1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
4 \\
11 \\
-5
\end{bmatrix}
\]

One of the three linear equations is

\[
\begin{align*}
x - 2y + z &= 4 \\
x + y + 3z &= 11 \\
2x - y &= -5
\end{align*}
\]

\[
\begin{align*}
x + 3z &= 11 \\
3y - z &= -5
\end{align*}
\]

Question 4

Each of the following four matrix equations represents a system of simultaneous linear equations.

\[
\begin{align*}
\begin{bmatrix}
1 & 3 \\
0 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} &=
\begin{bmatrix}
4 \\
8
\end{bmatrix} \\
\begin{bmatrix}
1 & 1 \\
2 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} &=
\begin{bmatrix}
5 \\
3
\end{bmatrix} \\
\begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} &=
\begin{bmatrix}
4 \\
8
\end{bmatrix} \\
\begin{bmatrix}
0 & 3 \\
0 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} &=
\begin{bmatrix}
6 \\
12
\end{bmatrix}
\end{align*}
\]

How many of these systems of simultaneous linear equations have a unique solution?

\[
\begin{align*}
A & \quad 0 \\
B & \quad 1 \\
C & \quad 2 \\
D & \quad 3 \\
E & \quad 4
\end{align*}
\]

[VCAA 2009 1MQ4]  
[VCAA 2008 1MQ6]  
[VCAA 2010 1MQ5]  
[VCAA 2011 1MQ3]
**Question 5**

Consider the following system of three simultaneous linear equations.

\[
\begin{align*}
2x + z &= 5 \\
x - 2y &= 0 \\
y - z &= -1
\end{align*}
\]

This system of equations can be written in matrix form as

\[
\begin{bmatrix}
2 & 1 & 5 \\
1 & -2 & 0 \\
1 & -1 & -1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
=
\begin{bmatrix}
5 \\
0 \\
-1
\end{bmatrix}
\]

**Question 6**

\[
\begin{align*}
2.8x + 0.7y &= 10 \\
1.4x + ky &= 6
\end{align*}
\]

The set of simultaneous linear equations above does not have a solution if \( k \) equals

A. -0.35  
B. -0.250  
C. 0  
D. 0.25  
E. 0.35

**Question 7**

How many of the following five sets of simultaneous linear equations have a unique solution?

\[
\begin{align*}
4x + 2y &= 10 \\
x &= 0 \\
x - y &= 3 \\
2x + y &= 5 \\
x &= 8 \\
2x + y &= 6 \\
x + y &= 3 \\
2x + y &= 10 \\
y &= 2
\end{align*}
\]

A. 1  
B. 2  
C. 3  
D. 4  
E. 5

**Question 8**

A worker can assemble 10 bookcases and four desks in 360 minutes, and eight bookcases and three desks in 280 minutes. If each bookcase takes \( b \) minutes to assemble and each desk takes \( d \) minutes to assemble, the matrix \( \begin{bmatrix} b & d \end{bmatrix} \) will be given by

\[
\begin{bmatrix}
-1.5 & 2 \\
4 & -5
\end{bmatrix}
\begin{bmatrix}
360 \\
280
\end{bmatrix}
\]

\[
\begin{bmatrix}
10 & 4 \\
8 & 3
\end{bmatrix}
\begin{bmatrix}
360 \\
280
\end{bmatrix}
\]

\[
\begin{bmatrix}
3 & -4 \\
-8 & 10
\end{bmatrix}
\begin{bmatrix}
360 \\
280
\end{bmatrix}
\]

\[
\begin{bmatrix}
5 & -2 \\
-4 & 1.5
\end{bmatrix}
\begin{bmatrix}
360 \\
280
\end{bmatrix}
\]

\[
\begin{bmatrix}
10 & 4 \\
360 & 280
\end{bmatrix}
\]

\[
\begin{bmatrix}
10 & 4 \\
360 & 280
\end{bmatrix}
\]
**Question 9**

The solution of the simultaneous equations (on the right) is given by

\[
\begin{align*}
2x + y + 2z &= 15 \\
x + z &= 6 \\
2y + z &= 8
\end{align*}
\]

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
-3 & -1 & 2 \\
-2 & 0 & 1 \\
4 & 1 & -2
\end{bmatrix}
\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}
\]

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 1 \\
0 & 2 & 1 \\
2 & 1 & 2
\end{bmatrix}
\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}
\]

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
-\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\
-1 & 1 & 0 \\
2 & -1 & 0
\end{bmatrix}
\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}
\]

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
-1 & -1 & 1 \\
-2 & 0 & 1 \\
2 & 1 & -1
\end{bmatrix}
\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}
\]

[VCAA 2012 1MQ3]

**Question 10**

Tickets for a school function are sold at the school office, the function hall and online. Different prices are charged for students, teachers and parents. The table shows the number of tickets sold at each place and the total value of sales.

<table>
<thead>
<tr>
<th></th>
<th>School office</th>
<th>Function hall</th>
<th>Online</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Student tickets</strong></td>
<td>283</td>
<td>35</td>
<td>84</td>
</tr>
<tr>
<td><strong>Teacher tickets</strong></td>
<td>28</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td><strong>Parent tickets</strong></td>
<td>5</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td><strong>Total sales</strong></td>
<td>$8712</td>
<td>$1143</td>
<td>$2609</td>
</tr>
</tbody>
</table>

For this function student tickets cost $x$, teacher tickets cost $y$, and parent tickets cost $z$.

a Use the information in the table to find the missing values in the shaded boxes of this matrix equation.

\[
\begin{bmatrix}
283 & 28 & 5 \\
84 & 3 & 7
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix} 8712 \\ 1143 \\ 2609 \end{bmatrix}
\]

1 mark

b Use the matrix equation to find the cost of a teacher ticket to the school function.

2 marks

[VCAA 2009 2MQ2]
Communication diagrams and matrices

Matrices are useful when investigating communications involving computer systems, friendship groups, social media, the military and travel. This communication diagram shows how four computers communicate with each other. The arrows indicate which way the communication goes. The communication between computer $A$ and computer $C$ is **two-way**: $A$ can communicate with $C$, and $C$ can communicate with $A$. However, the communication between $A$ and $D$ is **one-way**: $A$ can communicate with $D$, but $D$ can’t communicate with $A$.

The communication matrix $M$, which represents this communications diagram, is:

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$B$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$C$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$D$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

A communication matrix is a square matrix, where communication is indicated by a 1 and non-communication is indicated by a 0.

The leading diagonal of a communication matrix always consists of 0s since self-communication isn't considered communication in this context.
Exam hack
Communication matrices are sometimes written with the ‘From’ as the rows and the ‘To’ as the columns.

\[ M = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \]

Worked example 18
Find the communication matrix \( M \) that matches each of these communication diagrams.

**a**
Indicate communication by a 1 and non-communication by a 0.

\[ M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \]

**b**
Indicate communication by a 1 and non-communication by a 0.

\[ M = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \]
### Worked example 19

**a** The following communication matrix $M$ shows the direct flights for an airline that flies between four cities Arkton, Brenton, Corkton, and Dunton, indicated by their first letters. Copy and complete the communication diagram shown by drawing arrows between the letters to indicate direct flights.

$$
M = \begin{bmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{bmatrix}
$$

**b** Evaluate the matrix product $N = KM$, where $K = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$.

**c** What effect has pre-multiplying by $K$ had on $M$, and what information does the matrix $N$ contain?

**d** Which of the cities has the least direct flights to the other three cities?

---

**Working**

**a** Draw single arrows to indicate one-way communication/connection and double-ended arrows for two-way communication/connection.

**b** Multiply the matrices.

$$
N = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 & 2 \end{bmatrix}
$$

**c** Use the fact that $K$ is a summing matrix.

Pre-multiplying by $K$ has summed the elements in the columns in $M$. Matrix $N$ shows the total number of direct flights from each of the four cities to the other three cities.

**d** Look at the elements of $N$.

Matrix $N$ shows that there is only one direct flight from Brenton to the other cities, whereas the other cities each have two direct flights. So Brenton has the least number of direct flights to the other three cities.
Multi-step communication

Communication involves more than direct one-step communication. For example, in the communication diagram below, computers $A$ and $B$ don’t communicate directly, but $A$ can communicate with $D$, which can then communicate with $B$. This is called a two-step communication between $A$ and $B$.

Similarly, $B$ can’t communicate directly with $D$, but $B$ can communicate with $C$, which can communicate with $A$, which can communicate with $D$. This is called a three-step communication between $B$ and $D$.

For a communications matrix $M$:

$M^2$ gives the number of two-step communications,

$M^3$ gives the number of three-step communications,

$M^4$ gives the number of four-step communications, and so on.

$M + M^2$ gives us the number of one- or two-step communications,

$M + M^2 + M^3$ gives us the number of one-, two- or three-step communications, and so on.

Exam hack

Note the difference between two-way communication and two-step communication.

In this example:

- $A$ and $B$ have two-way communication
- $A \rightarrow B \rightarrow A$ is a two-step communication from $A$ to $A$.

In this example:

- There is no two-way communication.
- $A \rightarrow B \rightarrow C$ is a two-step communication from $A$ to $C$. 
Self-communication links

There is not much point in most communication situations where the sender and receiver are the same. These are called self-communication links. We included zeros in the leading diagonal of communication matrices so that we don't record one-step self-communication. However, once we start taking powers of communication matrices to find multi-step communication, non-zero elements appear in the leading diagonal.

For example, for

$$M = \begin{bmatrix}
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix} \quad \text{From} \quad A \quad \begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 2
\end{bmatrix} \quad \text{To} \quad A$$

From

$$0101$$

To

we get

$$M^2 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 2
\end{bmatrix} \quad \text{to} \quad A$$

The leading diagonal in $M^2$ shows the following self-communication links:

- one two-step communication from $A$ to $A$
- one two-step communication from $B$ to $B$
- two two-step communications from $D$ to $D$.

In most practical situations, self-communication should be ignored.

**Worked example 20**

For the communication diagram representing four computers and the matching matrix shown, find the following.

**a** i The two-step communication matrix.

ii The number of two-step self-communications from $A$ to $A$, listing all the paths.

iii The number of two-step communications from $A$ to $B$, listing all the paths.

**b** i The three-step communication matrix.

ii The number of three-step self-communications from $A$ to $A$, listing all the paths.

iii The number of three-step communications from $C$ to $B$, listing all the paths.

**c** i The matrix that shows the number of one- or two-step communications.

ii The number of one- or two-step communications from $D$ to $C$, listing all the paths.

iii Which communications can't be made with either one or two steps?

**d** i The matrix that shows the number of communications of no more than three steps.

ii Evidence that every one of the four computers can communicate with every other computer using one-, two- or three-step communications.
**Working**

\[
M^2 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

From
\[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
2 & 1 & 0 & 1 \\
1 & 0 & 2 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
C \\
D
\end{bmatrix}
\]

To
\[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
2 & 1 & 0 & 1 \\
1 & 0 & 2 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
C \\
D
\end{bmatrix}
\]

**a i** Find \(M^2\) using a CAS/calculator.

**ii** Find the number of two-step communications from \(M^2\), and find the paths from the communication diagram.

There is one two-step self-communication from \(A\) to \(A\).

\(A \rightarrow C \rightarrow A\)

**ii** Find the number of two-step communications from \(M^2\), and find the paths from the communication diagram.

There are two two-step communications from \(A\) to \(B\).

\(A \rightarrow D \rightarrow B\)

\(A \rightarrow C \rightarrow B\)

**b i** Find \(M^3\) using a CAS/calculator.

**ii** Find the number of three-step communications from \(M^3\), and find the paths from the communication diagram.

There is one three-step self-communication from \(A\) to \(A\).

\(A \rightarrow D \rightarrow C \rightarrow A\)

**ii** Find the number of three-step communications from \(M^3\), and find the paths from the communication diagram.

There are three three-step communications from \(C\) to \(B\).

\(C \rightarrow A \rightarrow D \rightarrow B\)

\(C \rightarrow A \rightarrow C \rightarrow B\)

\(C \rightarrow B \rightarrow C \rightarrow B\)
c  i  Find \( M + M^2 \).

\[
M + M^2 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
1 & 1 & 0 & 1 \\
2 & 1 & 0 & 1 \\
1 & 0 & 2 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 1 & 1 & 2 \\
2 & 1 & 2 & 2 \\
1 & 0 & 1 & 0
\end{bmatrix}
\]

From
\[
A \quad B \quad C \quad D
\]

\[
= \begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 1 & 1 & 2 \\
2 & 1 & 2 & 2 \\
1 & 0 & 1 & 0
\end{bmatrix}
\]

\[A \rightarrow B \rightarrow C\]

\[D \rightarrow C\]

ii  Find the number of one- or two-step communications from \( M + M^2 \), and find the paths from the communication diagram.

iii  Find the zeros in \( M + M^2 \).

B can't communicate with D in one or two steps.

D can't communicate with D in one or two steps.

d  i  ‘No more than three steps’ means ‘one-, two- or three-steps’, so find \( M + M^2 + M^3 \).

\[
M + M^2 + M^3 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 1 & 1 & 2 \\
2 & 1 & 2 & 2 \\
1 & 0 & 1 & 0
\end{bmatrix} + \begin{bmatrix}
1 & 0 & 2 & 1 \\
1 & 0 & 3 & 1 \\
3 & 2 & 1 & 2 \\
1 & 1 & 0 & 1
\end{bmatrix}
\]

From
\[
A \quad B \quad C \quad D
\]

\[
= \begin{bmatrix}
2 & 1 & 3 & 2 \\
3 & 1 & 4 & 3 \\
5 & 3 & 3 & 4 \\
2 & 1 & 1 & 1
\end{bmatrix}
\]

\[A \rightarrow B \rightarrow C \rightarrow D\]

There are no zeros in \( M + M^2 + M^3 \), so every one of the four computers can communicate with every other computer using one-, two- or three-step communications.

ii  Look for zeros in \( M + M^2 + M^3 \).
Communication matrices

Prep 1  WORKED EXAMPLE 18

Find the communication matrix $M$ that matches each of these communication diagrams.

a

b

Prep 2  WORKED EXAMPLE 19

a  The communication matrix $M$ shows the direct shuttles for a bus company that shuttles between the Art Gallery, the Beach, Central Station and Docklands, indicated by their first letters. Copy and complete the communication diagram below by drawing arrows between the letters to indicate direct shuttles.

b  Evaluate the matrix product $N = KM$, where

$$K = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}.$$

c  What effect has pre-multiplying by $K$ had on $M$, and what information does the matrix $N$ contain?

d  Which of the locations has the most direct shuttles to the other three locations?

Prep 3  WORKED EXAMPLE 20

For the communication diagram and matrix representing four ships in communication range, find the following.

a  i  The two-step communication matrix.

ii  The number of two-step self-communications from $A$ to $A$, listing all the paths.

iii  The number of two-step communications from $A$ to $B$, listing all the paths.

b  i  The three-step communication matrix.

ii  The number of three-step self-communications from $A$ to $A$, listing all the paths.

iii  The number of three-step communications from $C$ to $A$, listing all the paths.
The matrix that shows the number of one- or two-step communications.

ii The number of one- or two-step communications from A to C, listing all the paths.

iii Which communications can’t be made with either one or two steps?

d i The matrix that shows the number of communications of no more than three steps.

ii Evidence that every one of the four ships can communicate with every other ship using one-, two- or three-step communications.

**Communication matrices**

**Question 1**

Which one of the following could be a one-step communication matrix?

- **A**
  
  \[
  \begin{bmatrix}
  0 & 1 & 0 \\
  1 & 0 & 0 \\
  1 & 0 & 1 \\
  \end{bmatrix}
  \]

- **B**
  
  \[
  \begin{bmatrix}
  0 & 1 & 0 \\
  1 & 0 & 0 \\
  1 & 0 & 0 \\
  \end{bmatrix}
  \]

- **C**
  
  \[
  \begin{bmatrix}
  0 & 1 & 0 \\
  1 & 0 & 0 \\
  2 & 0 & 0 \\
  \end{bmatrix}
  \]

- **D**
  
  \[
  \begin{bmatrix}
  1 & 1 & 1 \\
  1 & 0 & 0 \\
  1 & 0 & 0 \\
  \end{bmatrix}
  \]

- **E**
  
  \[
  \begin{bmatrix}
  0 & 1 \\
  1 & 0 \\
  1 & 0 \\
  \end{bmatrix}
  \]

**Question 2**

Which of the statements regarding the following communication diagram is not true?

- **A** A and C have two-way communication.
- **B** There is two-step communication from A to B.
- **C** B and C have two-way communication.
- **D** A and D have two-way communication.
- **E** There is two-step communication from D to A.
**Question 3**

Which of the matrices represent the communication diagram?

From

A

\[
M = \begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

To

A

\[
M = \begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

B

\[
M = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

To

B

\[
M = \begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0
\end{bmatrix}
\]

C

\[
M = \begin{bmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

To

C

\[
M = \begin{bmatrix}
2 & 2 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 2 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}
\]

D

\[
M = \begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0
\end{bmatrix}
\]

To

D

\[
M = \begin{bmatrix}
2 & 2 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 2 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}
\]

E

\[
M = \begin{bmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

To

E

**Question 4**

Given the following matrix \( M^2 \) is a two-step communication matrix, which of the statements is false?

A  There is one two-step communication from C to D.
B  There are two two-step communications from B to A.
C  There are two two-step self-communications from C to C.
D  There is one two-step communication from A to C.
E  There are no two-step communications from B to D.
### Question 5

The diagram shows the tracks directly linking four camping sites $P$, $Q$, $R$ and $S$ in a national park. The shortest time that it takes to walk between the camping sites (in minutes), along each of these tracks, is also shown. A matrix that could be used to present the same information is

<table>
<thead>
<tr>
<th></th>
<th>$P$</th>
<th>$Q$</th>
<th>$R$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$Q$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$R$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$S$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

### Question 6

$A$, $B$, $C$, $D$ and $E$ are five intersections joined by roads as shown in the diagram. Some of these roads are one-way only.

The matrix indicates the direction that cars can travel along each of these roads. In this matrix:

- 1 in column $A$ and row $B$ indicates that cars can travel directly from $A$ to $B$
- 0 in column $B$ and row $A$ indicates that cars cannot travel directly from $B$ to $A$ (either it is a one-way road or no road exists).

From intersection

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
<th>$E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$B$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$C$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$D$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$E$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Cars can travel in both directions between intersections

- $A$ and $D$
- $B$ and $C$
- $C$ and $D$
- $D$ and $E$
- $C$ and $E$
**Question 7**

Matrix $F$ shows the flight connections for an airline that serves four cities, Anvil ($A$), Berga ($B$), Cantor ($C$) and Dantel ($D$).

In this matrix, the ‘1’ in column $C$ row $B$, for example, indicates that, using this airline, you can fly directly from Cantor to Berga.

The ‘0’ in column $C$ row $D$, for example, indicates that you cannot fly directly from Cantor to Dantel.

a. Copy and complete the following sentence.

On this airline, you can fly directly from Berga to [ ] and [ ]. 1 mark

b. List the route that you must follow to fly from Anvil to Cantor. 1 mark

c. Evaluate the matrix product $G = KF$, where $K = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. 1 mark

d. In the context of the problem, what information does matrix $G$ contain? 1 mark

[VCAA 2012 2MQ1]

**Question 8**

Five trout-breeding ponds, $P$, $Q$, $R$, $X$ and $V$, are connected by pipes, as shown in the diagram. The matrix $W$ is used to represent the information in this diagram.

In matrix $W$

- the 1 in column 1, row 2, for example, indicates that a pipe directly connects pond $P$ and pond $Q$
- the 0 in column 1, row 5, for example, indicates that pond $P$ and pond $V$ are not directly connected by a pipe.

a. Find the sum of the elements in row 3 of matrix $W$. 1 mark

b. In terms of the breeding ponds described, what does the sum of the elements in row 3 of matrix $W$ represent? 1 mark

The pipes connecting pond $P$ to pond $R$ and pond $P$ to pond $X$ are removed. Matrix $N$ will be used to show this situation. However, it has missing elements.

c. Copy and complete matrix $N$ by filling in the missing elements in row 1 and column 1. 1 mark

[VCAA 2013 2MQ1]
Matrix representation

- Matrices are usually named using capital letters.
- The order of a matrix is written as \( m \times n \), and we say ‘\( m \) by \( n \)’, where \( m \) is the number of rows and \( n \) is the number of columns.
- Each value in a matrix is called an element. \( a_{ij} \) is an element of matrix \( A \) where \( i \) is the row number and \( j \) is the column number.
- An \( m \times n \) matrix has \( mn \) elements.

Types of matrices

<table>
<thead>
<tr>
<th>Type of matrix</th>
<th>Description</th>
<th>Example</th>
</tr>
</thead>
</table>
| Row matrix     | A matrix with just one row. | \[
\begin{bmatrix}
-2 & 3 & 11 & 5
\end{bmatrix}
\] |
| Column matrix  | A matrix with just one column. | \[
\begin{bmatrix}
12 \\
-1 \\
0 \\
5
\end{bmatrix}
\] |
| Zero matrix    | A matrix where all the elements are 0. | \[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\] |
| Square matrix  | A matrix that has the same number of rows as columns. | \[
\begin{bmatrix}
0 & 9 \\
5 & 3
\end{bmatrix}
\] |
| Diagonal matrix| A square matrix where the only non-zero elements are in the leading diagonal. | \[
\begin{bmatrix}
4 & 0 \\
0 & -7
\end{bmatrix}
\] |
| Identity matrix| A square matrix where all the elements in the leading diagonal are 1 and the other elements are 0. | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] |
Addition and subtraction and scalar multiplication of matrices

- Addition and subtraction of matrices can only be performed with matrices of the same order.
- Addition/subtraction of matrices involves adding/subtracting the elements which are in corresponding positions in both matrices.
- When adding/subtracting two matrices, the result is a matrix of the same order as those being added/subtracted.
- When dealing with matrices, we use the word **scalar** to indicate a number that’s not in a matrix.
- **Scalar multiplication** involves multiplying every element in a matrix by the same number.

Matrix multiplication

- To multiply two matrices, $AB$, we multiply pairs of elements, working across the rows in $A$ and down the columns in $B$.
- Generally for matrices $AB \neq BA$.
- When we are multiplying two matrices $AB$, we say we are **pre-multiplying** by $A$ (as $A$ is before $B$) or **post-multiplying** by $B$ (as $B$ is after $A$).
- If matrix $A$ is of order $m \times n$ and matrix $B$ is of order $n \times p$, then the product $AB$ is defined, which means it’s possible. Otherwise $AB$ isn’t undefined, which means it’s not possible.

The number of rows in a defined matrix product is the number rows in the *first* matrix and the number of columns is the number of columns in the *last* matrix. If matrix $A$ is of order $m \times n$ and matrix $B$ is of order $n \times p$, then $AB$ will be of order $m \times p$.

- Only square matrices can be raised to a power.
- Any power of a matrix will always have the same order as the original matrix.
SUMMARY

Chapter 7: Matrices

SUMMARY

Summing matrices

- To sum the columns of a matrix, pre-multiply it by a suitable row matrix consisting of 1s.
- To sum the rows of a matrix, post-multiply it by a suitable column matrix consisting of 1s.

Inverse matrices

- The identity matrix, $I$, is the square matrix with leading diagonal elements 1 and other elements 0, which when multiplied by a square matrix, $A$, of the same order, leaves $A$ unchanged i.e., $AI = IA = A$.
- The inverse of a square matrix $A$, written as $A^{-1}$, is the matrix where $AA^{-1} = A^{-1}A = I$
- For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $\det(A) = ad - bc$
- The inverse of matrix $A$ is $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- Inverses are only defined for square matrices. The inverse doesn’t exist when $\det(A) = 0$.

Solving two simultaneous equations using matrices

- Matrices can be used to solve simultaneous equations.
- When solving simultaneous equations using matrices, always pre-multiply both sides by the inverse.
- When solving simultaneous equations using matrices, if there is no unique solution, a CAS/calculator will give you an error message.
Communication matrices

- A **communication diagram** shows one-way or two-way arrows between points indicating when communication occurs.

- A **communication matrix** is a square matrix where communication is indicated by a 1 and non-communication is indicated by a 0.

- The leading diagonal of a communication matrix always consists of 0s.

- **One-step communication** is direct communication between A and B.

- **One-way communication** is when A can communicate with B, but B can’t communicate with A.

- **Two-way communication** is when A can communicate with B, and B can communicate with A.

- An example of a **two-step communication** between A and B is $A \rightarrow D \rightarrow B$.

- An example of a **three-step communication** between A and B is $A \rightarrow D \rightarrow C \rightarrow B$.

- For a communications matrix $M$:
  - $M^2$ gives the number of two-step communications,
  - $M^3$ gives the number of three-step communications, and so on.
  - $M + M^2$ gives us the number of one- or two-step communications,
  - $M + M^2 + M^3$ gives us the number of one-, two- or three-step communications, and so on.

- **Self-communication links** are communications where the sender and receiver are the same.

- The leading diagonals in $M^2$, $M^3$, etc., show self-communication links.